Pinned group summary: SL-n3

Pinning-Sympy automated output

July 18, 2026

Contents

1 Basic group attributes

Group	SL-n3
Matrix size	3×3
Rank	2
Defining equations	$\det(X) = 1$

1.1 Lie algebra

The Lie algebra is denoted $\mathfrak{g}$. The defining equations for the $\mathfrak{g}$ are: $\mathrm{tr}(X) = 0$
A generic element of $\mathfrak{g}$ is:

$$\begin{bmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \\ x_{20} & x_{21} & -x_{00} - x_{11} \end{bmatrix}$$

1.2 Torus

The torus is denoted T. It has rank 2. A generic element of the T is

$$\begin{bmatrix} t_0 & 0 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & \frac{1}{t_0 t_1} \end{bmatrix}$$

A **character** of T is a group homomorphism from T to the multiplicative group of the ground field. The set of characters is denoted $X_*(T)$. In this document, elements of $X_*(T)$ are written as row vectors $\alpha = (a_1, a_2, \ldots, a_n)$ and such a vector is viewed as a function on diagonal matrices in the following way: for $\alpha = (a_1, a_2, \ldots, a_n) \in X_*(T)$ and $t = \mathrm{diag}(t_1, t_2, \ldots, t_n) \in T$,

$$\alpha(t) = \prod_{i=1}^{n} t_i^{a_i} = t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}$$

The rows of the matrix below are trivial characters, i.e. elements $\alpha \in X_*(T)$ which are the identity function on T. It would be more accurate to describe $X_*(T)$ as a quotient of a set of group homomorphisms modulo the lattice generated by trivial characters, but this distinction is not critical for our purposes.

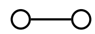
$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

2 Root system

Dynkin type	A_2
Irreducible	True
Reduced	True
Simply laced	True
Number of roots	6

2.1 Dynkin diagram

The root system is irreducible and has Dynkin diagram:



2.2 Cartan matrix

The root system is irreducible and has Cartan matrix:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

2.3 Roots

Definition 2.1. A root $\alpha \in \Phi$ is **multipliable** if $2\alpha \in \Phi$.

Root	**Short Form**	**Norm^2**	**Simple**	**Sign**	**Multipliable**
$(0,\ 1,\ -1)$	$(0,\ 1,\ -1)$	2	Yes	$+$	No
$(1,\ -1,\ 0)$	$(1,\ -1,\ 0)$	2	Yes	$+$	No
$(1,\ 0,\ -1)$	$(1,\ 0,\ -1)$	2	No	$+$	No
$(-1,\ 0,\ 1)$	$(-1,\ 0,\ 1)$	2	No	$-$	No
$(-1,\ 1,\ 0)$	$(-1,\ 1,\ 0)$	2	No	$-$	No
$(0,\ -1,\ 1)$	$(0,\ -1,\ 1)$	2	No	$-$	No

2.4 Coroots

Given a root system Φ, for each root $\alpha \in \Phi$, we find a coroot $\alpha^\vee$ (an integer vector of the same length) so that $\langle \alpha, \alpha^\vee \rangle = 2$. There is not necessarily a unique choice of $\alpha^\vee$, but for the purposes of this document, we choose $\alpha^\vee$ so that it is the integer vector of minimal Euclidean norm among those satisfying $\langle \alpha, \alpha^\vee \rangle = 2$.

Root	**Coroot**
$(-1,\ 0,\ 1)$	$(-1,\ 0,\ 1)$
$(-1,\ 1,\ 0)$	$(-1,\ 1,\ 0)$
$(0,\ -1,\ 1)$	$(0,\ -1,\ 1)$
$(0,\ 1,\ -1)$	$(0,\ 1,\ -1)$
$(1,\ -1,\ 0)$	$(1,\ -1,\ 0)$
$(1,\ 0,\ -1)$	$(1,\ 0,\ -1)$

2.5 Linear combinations of roots

Definition 2.2. For two roots α, β in a root system Φ, the set of positive integral linear combinations of α, β that are also roots is denoted $(\alpha, \beta) = \{i\alpha + j\beta \in \Phi : i, j \in \mathbb{Z}_{\geq 1}\}$.

α	β	Pairs (i, j)
$(-1,\ 0,\ 1)$	$(0,\ 1,\ -1)$	(1,1)
$(-1,\ 0,\ 1)$	$(1,\ -1,\ 0)$	(1,1)

α	β	Pairs (i, j)
$(-1,\ 1,\ 0)$	$(0,\ -1,\ 1)$	(1,1)
$(-1,\ 1,\ 0)$	$(1,\ 0,\ -1)$	(1,1)
$(0,\ -1,\ 1)$	$(1,\ 0,\ -1)$	(1,1)
$(0,\ 1,\ -1)$	$(1,\ -1,\ 0)$	(1,1)

3 Root spaces

3.1 Dimensions

For a root $\alpha \in \Phi$, the dimension of the associated root space V_α (and also the dimension of the root subgroup U_α) is denoted d_α. The table below lists these dimensions.

α	d_α
$(-1,\ 0,\ 1)$	1
$(-1,\ 1,\ 0)$	1
$(0,\ -1,\ 1)$	1
$(0,\ 1,\ -1)$	1
$(1,\ -1,\ 0)$	1
$(1,\ 0,\ -1)$	1

3.2 Root space table

The table below lists each root along with a generic element of the associated root space (inside the Lie algebra). These are computed as follows: given a character α, consider the set

$$V_\alpha = \left\{X \in \mathfrak{g} : t \cdot X \cdot t^{-1} = \alpha(t) \cdot X, \forall t \in T\right\}$$

This is a vector subspace of the Lie algebra $\mathfrak{g}$. When V_α is more than just the zero vector, α is called a **root** and the root system Φ is the set of all such roots. For this document, roots are computed as follows: compile a list of potential character vectors α, which are all possible integer vectors of length n with entries from $\{0, \pm 1, \pm 2\}$, then reduce that list of potentials by quotienting by the trivial characters of the torus. Next, instantiate a generic torus element t and a generic Lie algebra element X. For each potential character α, solve the equation

$$t \cdot X \cdot t^{-1} = \alpha(t) \cdot X$$

for X. If there is a nonzero solution space, add α to the set of roots.

Root	**Generic element of root space**
$(-1,\ 0,\ 1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u_0 & 0 & 0 \end{bmatrix}$
$(-1,\ 1,\ 0)$	$\begin{bmatrix} 0 & 0 & 0 \\ u_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$(0,\ -1,\ 1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & u_0 & 0 \end{bmatrix}$
$(0,\ 1,\ -1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & u_0 \\ 0 & 0 & 0 \end{bmatrix}$
$(1,\ -1,\ 0)$	$\begin{bmatrix} 0 & u_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$(1,\ 0,\ -1)$	$\begin{bmatrix} 0 & 0 & u_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4 Root subgroups

4.1 Root subgroup table

For the purposes of this document, root subgroups are found by simply exponentiating the associated root space. That is, for a root $\alpha \in \Phi$, the root space is

$$V_\alpha = \left\{X \in \mathfrak{g} : t \cdot X \cdot t^{-1} = \alpha(t) \cdot X, \forall t \in T\right\}$$

and the root subgroup is

$$U_\alpha = \left\{e^X = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots : X \in V_\alpha\right\}$$

For the purposes of the groups considered in this package, the X^3 and higher terms of the exponential series always vanish, and even the quadratic term frequently vanishes, so we can simplify the description of U_α to:

$$U_\alpha = \left\{1 + X + \frac{X^2}{2} : X \in V_\alpha\right\}$$

The table below lists each root along with a generic element of the associated root subgroup.

Root	**Generic element of root subgroup**
$(-1,\ 0,\ 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_0 & 0 & 1 \end{bmatrix}$
$(-1,\ 1,\ 0)$	$\begin{bmatrix} 1 & 0 & 0 \\ u_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$(0,\ -1,\ 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u_0 & 1 \end{bmatrix}$
$(0,\ 1,\ -1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & u_0 \\ 0 & 0 & 1 \end{bmatrix}$
$(1,\ -1,\ 0)$	$\begin{bmatrix} 1 & u_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$(1,\ 0,\ -1)$	$\begin{bmatrix} 1 & 0 & u_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4.2 Homomorphism property

For each root $\alpha \in \Phi$, the associated root subgroup map $X_\alpha : V_\alpha \to G$ is a pseudo-homomorphism with respect to the additive structure on V_α and the group operation (i.e. matrix multiplication) in G. That is, for any $u, v \in V_\alpha$,

$$X_\alpha(u) \cdot X_\alpha(v) = X_\alpha(u+v) \cdot \prod_{\substack{i>1 \\ i\alpha\in\Phi}} X_{i\alpha}\Big(q_\alpha^i(u,v)\Big) \tag{1}$$

for some homogeneous polynomial maps $q_\alpha^i : V_\alpha \times V_\alpha \to V_{i\alpha}$. Thus X_α is a group homomorphism if and only the product on the right hand side reduces to 1. The quantities $q_\alpha^i(u,v)$ are called **homomorphism defect coefficients**. For any root α of a root system Φ of any group considered in the Pinning-Sympy package, the set

$$\{i\alpha \in \Phi : i \in \mathbb{Z}_{\geq 2}\}$$

is either empty or a singleton set consisting of $\{2\alpha\}$. The non-empty case only arises when Φ is a non-reduced root system (of Dynkin type BC), and when α is a multipliable root.

Since the root system for this group is reduced, there are no multipliable roots and the product is always empty, hence each X_α is a group homomorphism there are no homomorphism defect coefficients.

5 Commutators

The commutator formula says that for two roots $\alpha, \beta \in \Phi$, the commutator of two root group elements $X_\alpha(u)$ and $X_\beta(v)$ is controlled by the structure of Φ, in particular by which positive integer linear combinations $i\alpha + j\beta$ are elements of Φ. Recall that

$$(\alpha, \beta) = \{i\alpha + j\beta \in \Phi : i, j \in \mathbb{Z}_{\geq 1}\}$$

Specifically, we have the commutator formula

$$\Big[X_\alpha(u), X_\beta(v)\Big] = \prod_{(\alpha,\beta)} X_{i\alpha+j\beta}\Big(N_{ij}^{\alpha\beta}(u,v)\Big)$$

for some homogeneous polynomial functions $N_{ij}^{\alpha\beta} : V_\alpha \times V_\beta \to V_{i\alpha+j\beta}$. We repeat here the table of linear combinations of roots. The quantity $N_{ij}^{\alpha\beta}(u,v)$ is called a **commutator coefficient**. The table below gives these coefficients.

α	β	i	j	$i\alpha+j\beta$	$N_{ij}^{\alpha\beta}(u,v)$
$(-1,\ 0,\ 1)$	$(0,\ 1,\ -1)$	1	1	$(-1,\ 1,\ 0)$	$-u_0v_0$
$(-1,\ 0,\ 1)$	$(1,\ -1,\ 0)$	1	1	$(0,\ -1,\ 1)$	u_0v_0
$(-1,\ 1,\ 0)$	$(0,\ -1,\ 1)$	1	1	$(-1,\ 0,\ 1)$	$-u_0v_0$
$(-1,\ 1,\ 0)$	$(1,\ 0,\ -1)$	1	1	$(0,\ 1,\ -1)$	u_0v_0
$(0,\ -1,\ 1)$	$(-1,\ 1,\ 0)$	1	1	$(-1,\ 0,\ 1)$	u_0v_0
$(0,\ -1,\ 1)$	$(1,\ 0,\ -1)$	1	1	$(1,\ -1,\ 0)$	$-u_0v_0$
$(0,\ 1,\ -1)$	$(-1,\ 0,\ 1)$	1	1	$(-1,\ 1,\ 0)$	u_0v_0
$(0,\ 1,\ -1)$	$(1,\ -1,\ 0)$	1	1	$(1,\ 0,\ -1)$	$-u_0v_0$
$(1,\ -1,\ 0)$	$(-1,\ 0,\ 1)$	1	1	$(0,\ -1,\ 1)$	$-u_0v_0$
$(1,\ -1,\ 0)$	$(0,\ 1,\ -1)$	1	1	$(1,\ 0,\ -1)$	u_0v_0
$(1,\ 0,\ -1)$	$(-1,\ 1,\ 0)$	1	1	$(0,\ 1,\ -1)$	$-u_0v_0$
$(1,\ 0,\ -1)$	$(0,\ -1,\ 1)$	1	1	$(1,\ -1,\ 0)$	u_0v_0

5.1 Commutator identities

$$\alpha = (-1,\ 0,\ 1)\,, \quad \beta = (0,\ 1,\ -1)$$

$$[X_\alpha(u_0), X_\beta(v_0)] = X_{\alpha+\beta}(-u_0v_0)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -u_0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -v_0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -u_0v_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = (-1,\ 0,\ 1)\,, \quad \beta = (1,\ -1,\ 0)$$

$$[X_\alpha(u_0), X_\beta(v_0)] = X_{\alpha+\beta}(u_0v_0)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & v_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -u_0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -v_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u_0v_0 & 1 \end{bmatrix}$$

$$\alpha = (-1,\ 1,\ 0)\,, \quad \beta = (0,\ -1,\ 1)$$

$$[X_\alpha(u_0), X_\beta(v_0)] = X_{\alpha+\beta}(-u_0v_0)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ u_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & v_0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -u_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -v_0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -u_0v_0 & 0 & 1 \end{bmatrix}$$

$$\alpha = (-1,\ 1,\ 0)\,, \quad \beta = (1,\ 0,\ -1)$$

$$[X_\alpha(u_0), X_\beta(v_0)] = X_{\alpha+\beta}(u_0 v_0)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ u_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & v_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -u_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -v_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & u_0 v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = (0,\ -1,\ 1)\,, \quad \beta = (-1,\ 1,\ 0)$$

$$[X_\alpha(u_0), X_\beta(v_0)] = X_{\alpha+\beta}(u_0 v_0)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u_0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ v_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -u_0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -v_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_0 v_0 & 0 & 1 \end{bmatrix}$$

$$\alpha = (0,\ -1,\ 1)\,, \quad \beta = (1,\ 0,\ -1)$$

$$[X_\alpha(u_0), X_\beta(v_0)] = X_{\alpha+\beta}(-u_0 v_0)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u_0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & v_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -u_0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -v_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -u_0 v_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = (0,\ 1,\ -1)\,, \quad \beta = (-1,\ 0,\ 1)$$

$$[X_\alpha(u_0), X_\beta(v_0)] = X_{\alpha+\beta}(u_0 v_0)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & u_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ v_0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -u_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -v_0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ u_0 v_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = (0,\ 1,\ -1)\,, \quad \beta = (1,\ -1,\ 0)$$

$$[X_\alpha(u_0), X_\beta(v_0)] = X_{\alpha+\beta}(-u_0 v_0)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & u_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & v_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -u_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -v_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -u_0 v_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = (1,\ -1,\ 0)\,, \quad \beta = (-1,\ 0,\ 1)$$

$$[X_\alpha(u_0), X_\beta(v_0)] = X_{\alpha+\beta}(-u_0 v_0)$$

$$\begin{bmatrix} 1 & u_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ v_0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -u_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -v_0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -u_0 v_0 & 1 \end{bmatrix}$$

$$\alpha = (1,\ -1,\ 0)\,, \quad \beta = (0,\ 1,\ -1)$$

$$[X_\alpha(u_0), X_\beta(v_0)] = X_{\alpha+\beta}(u_0 v_0)$$

$$\begin{bmatrix} 1 & u_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -u_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -v_0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & u_0 v_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = (1,\ 0,\ -1)\,, \quad \beta = (-1,\ 1,\ 0)$$

$$[X_\alpha(u_0), X_\beta(v_0)] = X_{\alpha+\beta}(-u_0 v_0)$$

$$\begin{bmatrix} 1 & 0 & u_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ v_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -u_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -v_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -u_0 v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = (1,\ 0,\ -1)\,, \quad \beta = (0,\ -1,\ 1)$$

$$[X_\alpha(u_0), X_\beta(v_0)] = X_{\alpha+\beta}(u_0 v_0)$$

$$\begin{bmatrix} 1 & 0 & u_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & v_0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -u_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -v_0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & u_0 v_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6 Weyl group

One of the defining characteristics of a root system Φ is that for any two roots $\alpha, \beta \in \Phi$, the reflection of β across the hyperplane perpendicular to α is another root. The reflection across the hyperplane perpendicular to α is denoted σ_α, and the previous sentence can be rephrased as saying that σ_α is a (bijective) function from Φ to itself. The reflection of β is then denoted $\sigma_\alpha(\beta)$. For each root $\alpha \in \Phi$, there is an element $w_\alpha \in G$ with the following properties:

1. $w_\alpha \in N_G(T)$, i.e. w_α normalizes the torus, i.e. $w_\alpha \cdot t \cdot w_\alpha^{-1} \in T$ for all $t \in T$.
2. $w_\alpha^2 \in Z_G(T)$, i.e. w_α^2 centralizes the torus, i.e. $w_\alpha^2 \cdot t \cdot (w_\alpha^2)^{-1} = t$ for all $t \in T$. Under some assumptions, $T = Z_G(T)$, though this is not true for all of the groups considered in this package.
3. For any root $\beta \in \Phi$, the conjugation of the root subgroup U_β by w_α is the root subgroup $U_{\sigma_\alpha(\beta)}$. That is, for $x \in U_\beta$,
$$w_\alpha \cdot x \cdot w_\alpha^{-1} \in U_{\sigma_\alpha}(\beta)$$
More precisely, given an element $x_\beta(u) \in U_\beta$, there is some function $\varphi_{\alpha\beta}(u)$ so that
$$w_\alpha \cdot x_\beta(u) \cdot w_\alpha^{-1} = x_{\sigma_\alpha(\beta)}\Big(\varphi_{\alpha\beta}(u)\Big)$$

Taken together, properies (1) and (2) above say that viewed as an element of the quotient group $N_G(T)/Z_G(T)$, (the class of) w_α is an element of order 2 (or order 1 but that doesn't happen for other reasons).

6.1 Weyl group elements

The table below lists roots along with an associated Weyl group element.

α	w_α
$(-1,\ 0,\ 1)$	$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
$(-1,\ 1,\ 0)$	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$(0,\ -1,\ 1)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
$(0,\ 1,\ -1)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
$(1,\ -1,\ 0)$	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$(1,\ 0,\ -1)$	$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

6.2 Weyl group conjugation coefficients

The table below lists roots along with the coefficients obtain in performing conjugation by an associated Weyl element.

α	β	$\gamma = \sigma_\alpha(\beta)$	$\varphi_{\alpha\beta}(u)$
$(-1,\ 0,\ 1)$	$(-1,\ 0,\ 1)$	$(1,\ 0,\ -1)$	$-u_0$
$(-1,\ 0,\ 1)$	$(-1,\ 1,\ 0)$	$(0,\ 1,\ -1)$	u_0
$(-1,\ 0,\ 1)$	$(0,\ -1,\ 1)$	$(1,\ -1,\ 0)$	$-u_0$
$(-1,\ 0,\ 1)$	$(0,\ 1,\ -1)$	$(-1,\ 1,\ 0)$	$-u_0$
$(-1,\ 0,\ 1)$	$(1,\ -1,\ 0)$	$(0,\ -1,\ 1)$	u_0
$(-1,\ 0,\ 1)$	$(1,\ 0,\ -1)$	$(-1,\ 0,\ 1)$	$-u_0$
$(-1,\ 1,\ 0)$	$(-1,\ 0,\ 1)$	$(0,\ -1,\ 1)$	$-u_0$
$(-1,\ 1,\ 0)$	$(-1,\ 1,\ 0)$	$(1,\ -1,\ 0)$	$-u_0$
$(-1,\ 1,\ 0)$	$(0,\ -1,\ 1)$	$(-1,\ 0,\ 1)$	u_0
$(-1,\ 1,\ 0)$	$(0,\ 1,\ -1)$	$(1,\ 0,\ -1)$	u_0
$(-1,\ 1,\ 0)$	$(1,\ -1,\ 0)$	$(-1,\ 1,\ 0)$	$-u_0$
$(-1,\ 1,\ 0)$	$(1,\ 0,\ -1)$	$(0,\ 1,\ -1)$	$-u_0$
$(0,\ -1,\ 1)$	$(-1,\ 0,\ 1)$	$(-1,\ 1,\ 0)$	$-u_0$
$(0,\ -1,\ 1)$	$(-1,\ 1,\ 0)$	$(-1,\ 0,\ 1)$	$-u_0$
$(0,\ -1,\ 1)$	$(0,\ -1,\ 1)$	$(0,\ 1,\ -1)$	u_0
$(0,\ -1,\ 1)$	$(0,\ 1,\ -1)$	$(0,\ -1,\ 1)$	u_0
$(0,\ -1,\ 1)$	$(1,\ -1,\ 0)$	$(1,\ 0,\ -1)$	$-u_0$
$(0,\ -1,\ 1)$	$(1,\ 0,\ -1)$	$(1,\ -1,\ 0)$	$-u_0$
$(0,\ 1,\ -1)$	$(-1,\ 0,\ 1)$	$(-1,\ 1,\ 0)$	$-u_0$
$(0,\ 1,\ -1)$	$(-1,\ 1,\ 0)$	$(-1,\ 0,\ 1)$	$-u_0$
$(0,\ 1,\ -1)$	$(0,\ -1,\ 1)$	$(0,\ 1,\ -1)$	u_0
$(0,\ 1,\ -1)$	$(0,\ 1,\ -1)$	$(0,\ -1,\ 1)$	u_0
$(0,\ 1,\ -1)$	$(1,\ -1,\ 0)$	$(1,\ 0,\ -1)$	$-u_0$
$(0,\ 1,\ -1)$	$(1,\ 0,\ -1)$	$(1,\ -1,\ 0)$	$-u_0$
$(1,\ -1,\ 0)$	$(-1,\ 0,\ 1)$	$(0,\ -1,\ 1)$	$-u_0$
$(1,\ -1,\ 0)$	$(-1,\ 1,\ 0)$	$(1,\ -1,\ 0)$	$-u_0$
$(1,\ -1,\ 0)$	$(0,\ -1,\ 1)$	$(-1,\ 0,\ 1)$	u_0
$(1,\ -1,\ 0)$	$(0,\ 1,\ -1)$	$(1,\ 0,\ -1)$	u_0
$(1,\ -1,\ 0)$	$(1,\ -1,\ 0)$	$(-1,\ 1,\ 0)$	$-u_0$
$(1,\ -1,\ 0)$	$(1,\ 0,\ -1)$	$(0,\ 1,\ -1)$	$-u_0$
$(1,\ 0,\ -1)$	$(-1,\ 0,\ 1)$	$(1,\ 0,\ -1)$	$-u_0$
$(1,\ 0,\ -1)$	$(-1,\ 1,\ 0)$	$(0,\ 1,\ -1)$	u_0
$(1,\ 0,\ -1)$	$(0,\ -1,\ 1)$	$(1,\ -1,\ 0)$	$-u_0$
$(1,\ 0,\ -1)$	$(0,\ 1,\ -1)$	$(-1,\ 1,\ 0)$	$-u_0$
$(1,\ 0,\ -1)$	$(1,\ -1,\ 0)$	$(0,\ -1,\ 1)$	u_0
$(1,\ 0,\ -1)$	$(1,\ 0,\ -1)$	$(-1,\ 0,\ 1)$	$-u_0$

6.3 Weyl group conjugation identities

$$\alpha = (-1,\ 0,\ 1)\,, \beta = (-1,\ 0,\ 1)\,, \sigma_\alpha(\beta) = (1,\ 0,\ -1)$$

$$w_\alpha X_\beta\,(u_0)\,w_\alpha^{-1} = X_{\sigma_\alpha(\beta)}\,(-u_0)$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -u_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = (-1,\ 0,\ 1)\,, \beta = (-1,\ 1,\ 0)\,, \sigma_\alpha(\beta) = (0,\ 1,\ -1)$$

$$w_\alpha X_\beta\,(u_0)\,w_\alpha^{-1} = X_{\sigma_\alpha(\beta)}\,(u_0)$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ u_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & u_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha=(-1,\ 0,\ 1)\,,\beta=(0,\ -1,\ 1)\,,\sigma_\alpha(\beta)=(1,\ -1,\ 0)$$

$$w_\alpha X_\beta\left(u_0\right)w_\alpha^{-1}=X_{\sigma_\alpha(\beta)}\left(-u_0\right)$$

$$\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&u_0&1\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}=\begin{bmatrix}1&-u_0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\alpha=(-1,\ 0,\ 1)\,,\beta=(0,\ 1,\ -1)\,,\sigma_\alpha(\beta)=(-1,\ 1,\ 0)$$

$$w_\alpha X_\beta\left(u_0\right)w_\alpha^{-1}=X_{\sigma_\alpha(\beta)}\left(-u_0\right)$$

$$\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&u_0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-u_0&1&0\\0&0&1\end{bmatrix}$$

$$\alpha=(-1,\ 0,\ 1)\,,\beta=(1,\ -1,\ 0)\,,\sigma_\alpha(\beta)=(0,\ -1,\ 1)$$

$$w_\alpha X_\beta\left(u_0\right)w_\alpha^{-1}=X_{\sigma_\alpha(\beta)}\left(u_0\right)$$

$$\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&u_0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&u_0&1\end{bmatrix}$$

$$\alpha=(-1,\ 0,\ 1)\,,\beta=(1,\ 0,\ -1)\,,\sigma_\alpha(\beta)=(-1,\ 0,\ 1)$$

$$w_\alpha X_\beta\left(u_0\right)w_\alpha^{-1}=X_{\sigma_\alpha(\beta)}\left(-u_0\right)$$

$$\begin{bmatrix}0&0&-1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&0&u_0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\-u_0&0&1\end{bmatrix}$$

$$\alpha=(-1,\ 1,\ 0)\,,\beta=(-1,\ 0,\ 1)\,,\sigma_\alpha(\beta)=(0,\ -1,\ 1)$$

$$w_\alpha X_\beta\left(u_0\right)w_\alpha^{-1}=X_{\sigma_\alpha(\beta)}\left(-u_0\right)$$

$$\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\u_0&0&1\end{bmatrix}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&-u_0&1\end{bmatrix}$$

$$\alpha=(-1,\ 1,\ 0)\,,\beta=(-1,\ 1,\ 0)\,,\sigma_\alpha(\beta)=(1,\ -1,\ 0)$$

$$w_\alpha X_\beta\left(u_0\right)w_\alpha^{-1}=X_{\sigma_\alpha(\beta)}\left(-u_0\right)$$

$$\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\u_0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&-u_0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\alpha=(-1,\ 1,\ 0)\,,\beta=(0,\ -1,\ 1)\,,\sigma_\alpha(\beta)=(-1,\ 0,\ 1)$$

$$w_\alpha X_\beta\left(u_0\right)w_\alpha^{-1}=X_{\sigma_\alpha(\beta)}\left(u_0\right)$$

$$\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&u_0&1\end{bmatrix}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\u_0&0&1\end{bmatrix}$$

$$\alpha=(-1,\ 1,\ 0)\,,\beta=(0,\ 1,\ -1)\,,\sigma_\alpha(\beta)=(1,\ 0,\ -1)$$

$$w_\alpha X_\beta\left(u_0\right)w_\alpha^{-1}=X_{\sigma_\alpha(\beta)}\left(u_0\right)$$

$$\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&u_0\\0&0&1\end{bmatrix}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&u_0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\alpha=(-1,\ 1,\ 0)\,,\beta=(1,\ -1,\ 0)\,,\sigma_\alpha(\beta)=(-1,\ 1,\ 0)$$

$$w_\alpha X_\beta\left(u_0\right) w_\alpha^{-1} = X_{\sigma_\alpha(\beta)}\left(-u_0\right)$$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & u_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -u_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = \left(-1,\ 1,\ 0\right), \beta = \left(1,\ 0,\ -1\right), \sigma_\alpha(\beta) = \left(0,\ 1,\ -1\right)$$

$$w_\alpha X_\beta\left(u_0\right) w_\alpha^{-1} = X_{\sigma_\alpha(\beta)}\left(-u_0\right)$$

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & u_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -u_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = \left(0,\ -1,\ 1\right), \beta = \left(-1,\ 0,\ 1\right), \sigma_\alpha(\beta) = \left(-1,\ 1,\ 0\right)$$

$$w_\alpha X_\beta\left(u_0\right) w_\alpha^{-1} = X_{\sigma_\alpha(\beta)}\left(-u_0\right)$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -u_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = \left(0,\ -1,\ 1\right), \beta = \left(-1,\ 1,\ 0\right), \sigma_\alpha(\beta) = \left(-1,\ 0,\ 1\right)$$

$$w_\alpha X_\beta\left(u_0\right) w_\alpha^{-1} = X_{\sigma_\alpha(\beta)}\left(-u_0\right)$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ u_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -u_0 & 0 & 1 \end{bmatrix}$$

$$\alpha = \left(0,\ -1,\ 1\right), \beta = \left(0,\ -1,\ 1\right), \sigma_\alpha(\beta) = \left(0,\ 1,\ -1\right)$$

$$w_\alpha X_\beta\left(u_0\right) w_\alpha^{-1} = X_{\sigma_\alpha(\beta)}\left(u_0\right)$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u_0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & u_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = \left(0,\ -1,\ 1\right), \beta = \left(0,\ 1,\ -1\right), \sigma_\alpha(\beta) = \left(0,\ -1,\ 1\right)$$

$$w_\alpha X_\beta\left(u_0\right) w_\alpha^{-1} = X_{\sigma_\alpha(\beta)}\left(u_0\right)$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & u_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u_0 & 1 \end{bmatrix}$$

$$\alpha = \left(0,\ -1,\ 1\right), \beta = \left(1,\ -1,\ 0\right), \sigma_\alpha(\beta) = \left(1,\ 0,\ -1\right)$$

$$w_\alpha X_\beta\left(u_0\right) w_\alpha^{-1} = X_{\sigma_\alpha(\beta)}\left(-u_0\right)$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & u_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -u_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = \left(0,\ -1,\ 1\right), \beta = \left(1,\ 0,\ -1\right), \sigma_\alpha(\beta) = \left(1,\ -1,\ 0\right)$$

$$w_\alpha X_\beta\left(u_0\right) w_\alpha^{-1} = X_{\sigma_\alpha(\beta)}\left(-u_0\right)$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & u_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -u_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$