

Pinned group summary: SU-n6-q2-eps-minus1

Pinning-Sympy automated output

July 18, 2026

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1 Basic group attributes

Group	SU-n6-q2-eps-minus1
Matrix size	6×6
Rank	2
Defining equations	$X^*HX = H$ and $\det(X) = 1$

1.1 Skew-hermitian form

Dimension	6
Witt index	2
Primitive element	$\sqrt{d}$
Epsilon	-1

Matrix:
$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0\sqrt{d} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_1\sqrt{d} \end{bmatrix}$$

1.2 Lie algebra

The Lie algebra is denoted $\mathfrak{g}$. The defining equations for the $\mathfrak{g}$ are: $X^*H + HX = 0$ and $\text{tr}(X) = 0$. A generic element of $\mathfrak{g}$ is:

$$\begin{bmatrix} \sqrt{d}x_{i00} + x_{r00} & \sqrt{d}x_{i01} + x_{r01} & -x_{r02} & -\sqrt{d}x_{i12} + x_{r12} & c_0\sqrt{d} \left(-\sqrt{d}x_{i42} + x_{r42} \right) & c_1\sqrt{d} \left(-\sqrt{d}x_{i52} + x_{r52} \right) \\ \sqrt{d}x_{i10} + x_{r10} & -\sqrt{d}x_{i00} - \frac{\sqrt{d}x_{i44}}{2} - \frac{\sqrt{d}x_{i55}}{2} + x_{r11} & \sqrt{d}x_{i12} + x_{r12} & -x_{r13} & c_0\sqrt{d} \left(-\sqrt{d}x_{i43} + x_{r43} \right) & c_1\sqrt{d} \left(-\sqrt{d}x_{i53} + x_{r53} \right) \\ -x_{r20} & -\sqrt{d}x_{i30} + x_{r30} & \sqrt{d}x_{i00} - x_{r00} & \sqrt{d}x_{i10} - x_{r10} & -c_0\sqrt{d} \left(-\sqrt{d}x_{i40} + x_{r40} \right) & -c_1\sqrt{d} \left(-\sqrt{d}x_{i50} + x_{r50} \right) \\ \sqrt{d}x_{i30} + x_{r30} & -x_{r31} & \sqrt{d}x_{i01} - x_{r01} & -\sqrt{d}x_{i00} - \frac{\sqrt{d}x_{i44}}{2} - \frac{\sqrt{d}x_{i55}}{2} - x_{r11} & -c_0\sqrt{d} \left(-\sqrt{d}x_{i41} + x_{r41} \right) & -c_1\sqrt{d} \left(-\sqrt{d}x_{i51} + x_{r51} \right) \\ \sqrt{d}x_{i40} + x_{r40} & \sqrt{d}x_{i41} + x_{r41} & \sqrt{d}x_{i42} + x_{r42} & \sqrt{d}x_{i43} + x_{r43} & \sqrt{d}x_{i44} & \sqrt{d}x_{i45} + x_{r45} \\ \sqrt{d}x_{i50} + x_{r50} & \sqrt{d}x_{i51} + x_{r51} & \sqrt{d}x_{i52} + x_{r52} & \sqrt{d}x_{i53} + x_{r53} & -\frac{c_0(-\sqrt{d}x_{i45} + x_{r45})}{c_1} & \sqrt{d}x_{i55} \end{bmatrix}$$

1.3 Torus

The torus is denoted T . It has rank 2. A generic element of the T is

$$\begin{bmatrix} t_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{t_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{t_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A **character** of T is a group homomorphism from T to the multiplicative group of the ground field. The set of characters is denoted $X_*(T)$. In this document, elements of $X_*(T)$ are written as row vectors $\alpha = (a_1, a_2, \dots, a_n)$ and such a vector is viewed as a function on diagonal matrices in the following way: for $\alpha = (a_1, a_2, \dots, a_n) \in X_*(T)$ and $t = \text{diag}(t_1, t_2, \dots, t_n) \in T$,

$$\alpha(t) = \prod_{i=1}^n t_i^{a_i} = t_1^{a_1} t_2^{a_2} \dots t_n^{a_n}$$

The rows of the matrix below are trivial characters, i.e. elements $\alpha \in X_*(T)$ which are the identity function on T . It would be more accurate to describe $X_*(T)$ as a quotient of a set of group homomorphisms modulo the lattice

generated by trivial characters, but this distinction is not critical for our purposes.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

2 Root system

Dynkin type	BC_2
Irreducible	True
Reduced	False
Simply laced	False
Number of roots	12

2.1 Dynkin diagram

The root system is irreducible and has Dynkin diagram:



2.2 Cartan matrix

The root system is irreducible and has Cartan matrix:

$$\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

2.3 Roots

Definition 2.1. A root $\alpha \in \Phi$ is **multipliable** if $2\alpha \in \Phi$.

Root	Short Form	Norm ²	Simple	Sign	Multipliable
(0, 1, 0, 0, 0, 0)	(0, 1)	1	Yes	+	Yes
(1, 0, 0, 0, 0, 0)	(1, 0)	1	No	+	Yes
(-1, 0, 0, 0, 0, 0)	(-1, 0)	1	No	-	Yes
(0, -1, 0, 0, 0, 0)	(0, -1)	1	No	-	Yes
(1, -1, 0, 0, 0, 0)	(1, -1)	2	Yes	+	No
(1, 1, 0, 0, 0, 0)	(1, 1)	2	No	+	No
(-1, -1, 0, 0, 0, 0)	(-1, -1)	2	No	-	No
(-1, 1, 0, 0, 0, 0)	(-1, 1)	2	No	-	No
(0, 2, 0, 0, 0, 0)	(0, 2)	4	No	+	No
(2, 0, 0, 0, 0, 0)	(2, 0)	4	No	+	No
(-2, 0, 0, 0, 0, 0)	(-2, 0)	4	No	-	No
(0, -2, 0, 0, 0, 0)	(0, -2)	4	No	-	No

2.4 Coroots

Given a root system Φ , for each root $\alpha \in \Phi$, we find a coroot $\alpha^\vee$ (an integer vector of the same length) so that $\langle \alpha, \alpha^\vee \rangle = 2$. There is not necessarily a unique choice of $\alpha^\vee$, but for the purposes of this document, we choose $\alpha^\vee$ so that it is the integer vector of minimal Euclidean norm among those satisfying $\langle \alpha, \alpha^\vee \rangle = 2$.

Root	Coroot
$(-2, 0, 0, 0, 0, 0)$	$(-1, 0, 1, 0, 0, 0)$
$(-1, -1, 0, 0, 0, 0)$	$(-1, -1, 1, 1, 0, 0)$
$(-1, 0, 0, 0, 0, 0)$	$(-2, 0, 2, 0, 0, 0)$
$(-1, 1, 0, 0, 0, 0)$	$(-1, 1, 1, -1, 0, 0)$
$(0, -2, 0, 0, 0, 0)$	$(0, -1, 0, 1, 0, 0)$
$(0, -1, 0, 0, 0, 0)$	$(0, -2, 0, 2, 0, 0)$
$(0, 1, 0, 0, 0, 0)$	$(0, 2, 0, -2, 0, 0)$
$(0, 2, 0, 0, 0, 0)$	$(0, 1, 0, -1, 0, 0)$
$(1, -1, 0, 0, 0, 0)$	$(1, -1, -1, 1, 0, 0)$
$(1, 0, 0, 0, 0, 0)$	$(2, 0, -2, 0, 0, 0)$
$(1, 1, 0, 0, 0, 0)$	$(1, 1, -1, -1, 0, 0)$
$(2, 0, 0, 0, 0, 0)$	$(1, 0, -1, 0, 0, 0)$

2.5 Linear combinations of roots

Definition 2.2. For two roots α, β in a root system Φ , the set of positive integral linear combinations of α, β that are also roots is denoted $(\alpha, \beta) = \{i\alpha + j\beta \in \Phi : i, j \in \mathbb{Z}_{\geq 1}\}$.

α	β	Pairs (i, j)
$(-1, -1)$	$(-1, 1)$	$(1, 1)$
$(-1, -1)$	$(1, -1)$	$(1, 1)$
$(-1, 0)$	$(0, -1)$	$(1, 1)$
$(-1, 0)$	$(0, 1)$	$(1, 1)$
$(-1, 1)$	$(1, 1)$	$(1, 1)$
$(0, -1)$	$(1, 0)$	$(1, 1)$
$(0, 1)$	$(1, 0)$	$(1, 1)$
$(1, -1)$	$(1, 1)$	$(1, 1)$
$(-2, 0)$	$(1, -1)$	$(1, 1), (1, 2)$
$(-2, 0)$	$(1, 1)$	$(1, 1), (1, 2)$
$(-1, -1)$	$(0, 2)$	$(1, 1), (2, 1)$
$(-1, -1)$	$(2, 0)$	$(1, 1), (2, 1)$
$(-1, 1)$	$(0, -2)$	$(1, 1), (2, 1)$
$(-1, 1)$	$(2, 0)$	$(1, 1), (2, 1)$
$(0, -2)$	$(1, 1)$	$(1, 1), (1, 2)$
$(0, 2)$	$(1, -1)$	$(1, 1), (1, 2)$
$(-1, -1)$	$(0, 1)$	$(1, 1), (1, 2), (2, 2)$
$(-1, -1)$	$(1, 0)$	$(1, 1), (1, 2), (2, 2)$
$(-1, 0)$	$(1, -1)$	$(1, 1), (2, 1), (2, 2)$
$(-1, 0)$	$(1, 1)$	$(1, 1), (2, 1), (2, 2)$
$(-1, 1)$	$(0, -1)$	$(1, 1), (1, 2), (2, 2)$
$(-1, 1)$	$(1, 0)$	$(1, 1), (1, 2), (2, 2)$
$(0, -1)$	$(1, 1)$	$(1, 1), (2, 1), (2, 2)$
$(0, 1)$	$(1, -1)$	$(1, 1), (2, 1), (2, 2)$

3 Root spaces

3.1 Dimensions

For a root $\alpha \in \Phi$, the dimension of the associated root space V_α (and also the dimension of the root subgroup U_α) is denoted d_α . The table below lists these dimensions.

α	d_α
$(-2, 0, 0, 0, 0, 0)$	1
$(0, -2, 0, 0, 0, 0)$	1
$(0, 2, 0, 0, 0, 0)$	1
$(2, 0, 0, 0, 0, 0)$	1
$(-1, -1, 0, 0, 0, 0)$	2
$(-1, 1, 0, 0, 0, 0)$	2
$(1, -1, 0, 0, 0, 0)$	2
$(1, 1, 0, 0, 0, 0)$	2
$(-1, 0, 0, 0, 0, 0)$	4
$(0, -1, 0, 0, 0, 0)$	4
$(0, 1, 0, 0, 0, 0)$	4
$(1, 0, 0, 0, 0, 0)$	4

3.2 Root space table

The table below lists each root along with a generic element of the associated root space (inside the Lie algebra). These are computed as follows: given a character α , consider the set

$$V_\alpha = \{X \in \mathfrak{g} : t \cdot X \cdot t^{-1} = \alpha(t) \cdot X, \forall t \in T\}$$

This is a vector subspace of the Lie algebra $\mathfrak{g}$. When V_α is more than just the zero vector, α is called a **root** and the root system Φ is the set of all such roots. For this document, roots are computed as follows: compile a list of potential character vectors α , which are all possible integer vectors of length n with entries from $\{0, \pm 1, \pm 2\}$, then reduce that list of potentials by quotienting by the trivial characters of the torus. Next, instantiate a generic torus element t and a generic Lie algebra element X . For each potential character α , solve the equation

$$t \cdot X \cdot t^{-1} = \alpha(t) \cdot X$$

for X . If there is a nonzero solution space, add α to the set of roots.

Root	Generic element of root space
$(-2, 0, 0, 0, 0, 0)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -u_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$(-1, -1, 0, 0, 0, 0)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{d}u_1 + u_0 & 0 & 0 & 0 & 0 \\ \sqrt{d}u_1 + u_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Root	Generic element of root space
$(-1, 0, 0, 0, 0, 0)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0(-\sqrt{d}u_2 + du_0) & c_1(-\sqrt{d}u_1 + du_3) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{d}u_0 + u_2 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{d}u_3 + u_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$(-1, 1, 0, 0, 0, 0)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{d}u_1 + u_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{d}u_1 - u_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$(0, -2, 0, 0, 0, 0)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -u_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$(0, -1, 0, 0, 0, 0)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0(-\sqrt{d}u_0 + du_2) & c_1(-\sqrt{d}u_1 + du_3) \\ 0 & \sqrt{d}u_2 + u_0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{d}u_3 + u_1 & 0 & 0 & 0 & 0 \end{bmatrix}$
$(0, 1, 0, 0, 0, 0)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0(\sqrt{d}u_2 - du_3) & c_1(\sqrt{d}u_1 - du_0) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{d}u_3 + u_2 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{d}u_0 + u_1 & 0 & 0 \end{bmatrix}$
$(0, 2, 0, 0, 0, 0)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -u_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$(1, -1, 0, 0, 0, 0)$	$\begin{bmatrix} 0 & \sqrt{d}u_1 + u_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{d}u_1 - u_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
$(1, 0, 0, 0, 0, 0)$	$\begin{bmatrix} 0 & 0 & 0 & 0 & c_0(\sqrt{d}u_0 - du_1) & c_1(\sqrt{d}u_2 - du_3) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{d}u_1 + u_0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{d}u_3 + u_2 & 0 & 0 & 0 \end{bmatrix}$

Root	Generic element of root space						
(1, 1, 0, 0, 0, 0)	$\begin{bmatrix} 0 & 0 & 0 & -\sqrt{d}u_0 + u_1 & 0 & 0 \\ 0 & 0 & \sqrt{d}u_0 + u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$						
(2, 0, 0, 0, 0, 0)	$\begin{bmatrix} 0 & 0 & -u_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$						

4 Root subgroups

4.1 Root subgroup table

For the purposes of this document, root subgroups are found by simply exponentiating the associated root space. That is, for a root $\alpha \in \Phi$, the root space is

$$V_\alpha = \{X \in \mathfrak{g} : t \cdot X \cdot t^{-1} = \alpha(t) \cdot X, \forall t \in T\}$$

and the root subgroup is

$$U_\alpha = \left\{ e^X = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots : X \in V_\alpha \right\}$$

For the purposes of the groups considered in this package, the X^3 and higher terms of the exponential series always vanish, and even the quadratic term frequently vanishes, so we can simplify the description of U_α to:

$$U_\alpha = \left\{ 1 + X + \frac{X^2}{2} : X \in V_\alpha \right\}$$

The table below lists each root along with a generic element of the associated root subgroup.

Root	Generic element of root subgroup
$(-2, 0, 0, 0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -u_0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
$(-1, -1, 0, 0, 0, 0)$	$\begin{bmatrix} 1 & & 0 & & 0 & 0 & 0 & 0 \\ 0 & & 1 & & 0 & 0 & 0 & 0 \\ 0 & & -\sqrt{d}u_1 + u_0 & & 1 & 0 & 0 & 0 \\ \sqrt{d}u_1 + u_0 & & 0 & & 0 & 1 & 0 & 0 \\ 0 & & 0 & & 0 & 0 & 1 & 0 \\ 0 & & 0 & & 0 & 0 & 0 & 1 \end{bmatrix}$
$(-1, 0, 0, 0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{d}(c_0du_0^2 - c_0u_2^2 + c_1du_3^2 - c_1u_1^2)}{2} & 0 & 1 & 0 & c_0(-\sqrt{d}u_2 + du_0) & c_1(-\sqrt{d}u_1 + du_3) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \sqrt{d}u_0 + u_2 & 0 & 0 & 0 & 1 & 0 \\ \sqrt{d}u_3 + u_1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
$(-1, 1, 0, 0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{d}u_1 + u_0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{d}u_1 - u_0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
$(0, -2, 0, 0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -u_0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Root	Generic element of root subgroup
$(0, -1, 0, 0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{d}(c_0 du_2^2 - c_0 u_0^2 + c_1 du_3^2 - c_1 u_1^2)}{2} & 0 & 1 & c_0(-\sqrt{d}u_0 + du_2) & c_1(-\sqrt{d}u_1 + du_3) \\ 0 & \sqrt{d}u_2 + u_0 & 0 & 0 & 1 & 0 \\ 0 & \sqrt{d}u_3 + u_1 & 0 & 0 & 0 & 1 \end{bmatrix}$
$(0, 1, 0, 0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{d}(-c_0 du_3^2 + c_0 u_2^2 - c_1 du_0^2 + c_1 u_1^2)}{2} & c_0(\sqrt{d}u_2 - du_3) & c_1(\sqrt{d}u_1 - du_0) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{d}u_3 + u_2 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{d}u_0 + u_1 & 0 & 1 \end{bmatrix}$
$(0, 2, 0, 0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -u_0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
$(1, -1, 0, 0, 0, 0)$	$\begin{bmatrix} 1 & \sqrt{d}u_1 + u_0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{d}u_1 - u_0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
$(1, 0, 0, 0, 0, 0)$	$\begin{bmatrix} 1 & 0 & \frac{\sqrt{d}(-c_0 du_1^2 + c_0 u_0^2 - c_1 du_3^2 + c_1 u_2^2)}{2} & 0 & c_0(\sqrt{d}u_0 - du_1) & c_1(\sqrt{d}u_2 - du_3) \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{d}u_1 + u_0 & 0 & 1 & 0 \\ 0 & 0 & \sqrt{d}u_3 + u_2 & 0 & 0 & 1 \end{bmatrix}$
$(1, 1, 0, 0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 & -\sqrt{d}u_0 + u_1 & 0 & 0 \\ 0 & 1 & \sqrt{d}u_0 + u_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
$(2, 0, 0, 0, 0, 0)$	$\begin{bmatrix} 1 & 0 & -u_0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

4.2 Homomorphism property

For each root $\alpha \in \Phi$, the associated root subgroup map $X_\alpha : V_\alpha \rightarrow G$ is a pseudo-homomorphism with respect to the additive structure on V_α and the group operation (i.e. matrix multiplication) in G . That is, for any $u, v \in V_\alpha$,

$$X_\alpha(u) \cdot X_\alpha(v) = X_\alpha(u + v) \cdot \prod_{\substack{i \geq 1 \\ i\alpha \in \Phi}} X_{i\alpha}(q_\alpha^i(u, v)) \quad (1)$$

for some homogeneous polynomial maps $q_\alpha^i : V_\alpha \times V_\alpha \rightarrow V_{i\alpha}$. Thus X_α is a group homomorphism if and only the product on the right hand side reduces to 1. The quantities $q_\alpha^i(u, v)$ are called **homomorphism defect coefficients**. For any root α of a root system Φ of any group considered in the Pinning-Sympy package, the set

$$\{i\alpha \in \Phi : i \in \mathbb{Z}_{\geq 2}\}$$

is either empty or a singleton set consisting of $\{2\alpha\}$. The non-empty case only arises when Φ is a non-reduced root system (of Dynkin type BC), and when α is a multipliable root.

The table below lists all multipliable roots and the associated homomorphism defect.

α	$q_\alpha^i(u, v)$
$(-1, 0)$	$-c_0 du_0 v_2 + c_0 du_2 v_0 + c_1 du_1 v_3 - c_1 du_3 v_1$
$(0, -1)$	$c_0 du_0 v_2 - c_0 du_2 v_0 + c_1 du_1 v_3 - c_1 du_3 v_1$
$(0, 1)$	$-c_0 du_2 v_3 + c_0 du_3 v_2 + c_1 du_0 v_1 - c_1 du_1 v_0$
$(1, 0)$	$-c_0 du_0 v_1 + c_0 du_1 v_0 - c_1 du_2 v_3 + c_1 du_3 v_2$

Here are the pseudo-homomorphism equations.

$$\alpha = (-1, 0, 0, 0, 0, 0)$$

$$X_\alpha \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} X_\alpha \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = X_\alpha \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot X_{2\alpha} (-c_0 du_0 v_2 + c_0 du_2 v_0 + c_1 du_1 v_3 - c_1 du_3 v_1)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = (0, -1, 0, 0, 0, 0)$$

$$X_\alpha \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} X_\alpha \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = X_\alpha \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot X_{2\alpha} (c_0 du_0 v_2 - c_0 du_2 v_0 + c_1 du_1 v_3 - c_1 du_3 v_1)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = (0, 1, 0, 0, 0, 0)$$

$$X_\alpha \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} X_\alpha \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = X_\alpha \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot X_{2\alpha} (-c_0 du_2 v_3 + c_0 du_3 v_2 + c_1 du_0 v_1 - c_1 du_1 v_0)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = (1, 0, 0, 0, 0, 0)$$

$$X_\alpha \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} X_\alpha \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = X_\alpha \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot X_{2\alpha} (-c_0 du_0 v_1 + c_0 du_1 v_0 - c_1 du_2 v_3 + c_1 du_3 v_2)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5 Commutators

The commutator formula says that for two roots $\alpha, \beta \in \Phi$, the commutator of two root group elements $X_\alpha(u)$ and $X_\beta(v)$ is controlled by the structure of Φ , in particular by which positive integer linear combinations $i\alpha + j\beta$ are elements of Φ . Recall that

$$(\alpha, \beta) = \{i\alpha + j\beta \in \Phi : i, j \in \mathbb{Z}_{\geq 1}\}$$

Specifically, we have the commutator formula

$$[X_\alpha(u), X_\beta(v)] = \prod_{(\alpha, \beta)} X_{i\alpha + j\beta} \left(N_{ij}^{\alpha\beta}(u, v) \right)$$

for some homogeneous polynomial functions $N_{ij}^{\alpha\beta} : V_\alpha \times V_\beta \rightarrow V_{i\alpha + j\beta}$. We repeat here the table of linear combinations of roots. The quantity $N_{ij}^{\alpha\beta}(u, v)$ is called a **commutator coefficient**. The table below gives these coefficients.

α	β	i	j	$i\alpha + j\beta$	$N_{ij}^{\alpha\beta}(u, v)$
$(-1, -1)$	$(-1, 1)$	1	1	$(-2, 0)$	$2du_1v_1 - 2u_0v_0$
$(-1, -1)$	$(1, -1)$	1	1	$(0, -2)$	$-2du_1v_1 - 2u_0v_0$
$(-1, 0)$	$(0, -1)$	1	1	$(-1, -1)$	$[d(c_0u_0v_0 - c_0u_2v_2 - c_1u_1v_3 + c_1u_3v_1) \quad -c_0du_0v_2 + c_0u_2v_0 - c_1du_3v_3 + c_1u_1v_1]$
$(-1, 0)$	$(0, 1)$	1	1	$(-1, 1)$	$[d(-c_0u_0v_2 + c_0u_2v_3 + c_1u_1v_0 - c_1u_3v_1) \quad c_0du_0v_3 - c_0u_2v_2 + c_1du_3v_0 - c_1u_1v_1]$
$(-1, 1)$	$(-1, -1)$	1	1	$(-2, 0)$	$-2du_1v_1 + 2u_0v_0$
$(-1, 1)$	$(1, 1)$	1	1	$(0, 2)$	$2du_1v_0 - 2u_0v_1$
$(0, -1)$	$(-1, 0)$	1	1	$(-1, -1)$	$[d(-c_0u_0v_0 + c_0u_2v_2 - c_1u_1v_3 + c_1u_3v_1) \quad c_0du_2v_0 - c_0u_0v_2 + c_1du_3v_3 - c_1u_1v_1]$
$(0, -1)$	$(1, 0)$	1	1	$(1, -1)$	$[d(c_0u_0v_1 - c_0u_2v_0 + c_1u_1v_3 - c_1u_3v_2) \quad c_0du_2v_1 - c_0u_0v_0 + c_1du_3v_3 - c_1u_1v_2]$
$(0, 1)$	$(-1, 0)$	1	1	$(-1, 1)$	$[d(c_0u_2v_0 - c_0u_3v_2 - c_1u_0v_1 + c_1u_1v_3) \quad -c_0du_3v_0 + c_0u_2v_2 - c_1du_0v_3 + c_1u_1v_1]$
$(0, 1)$	$(1, 0)$	1	1	$(1, 1)$	$[-c_0du_3v_1 + c_0u_2v_0 - c_1du_0v_3 + c_1u_1v_2 \quad d(c_0u_2v_1 - c_0u_3v_0 - c_1u_0v_2 + c_1u_1v_3)]$
$(1, -1)$	$(-1, -1)$	1	1	$(0, -2)$	$2du_1v_1 + 2u_0v_0$
$(1, -1)$	$(1, 1)$	1	1	$(2, 0)$	$-2du_1v_0 - 2u_0v_1$
$(1, 0)$	$(0, -1)$	1	1	$(1, -1)$	$[d(c_0u_0v_2 - c_0u_1v_0 + c_1u_2v_3 - c_1u_3v_1) \quad -c_0du_1v_2 + c_0u_0v_0 - c_1du_3v_3 + c_1u_2v_1]$
$(1, 0)$	$(0, 1)$	1	1	$(1, 1)$	$[c_0du_1v_3 - c_0u_0v_2 + c_1du_3v_0 - c_1u_2v_1 \quad d(c_0u_0v_3 - c_0u_1v_2 + c_1u_2v_0 - c_1u_3v_1)]$
$(1, 1)$	$(-1, 1)$	1	1	$(0, 2)$	$-2du_0v_1 + 2u_1v_0$
$(1, 1)$	$(1, -1)$	1	1	$(2, 0)$	$2du_0v_1 + 2u_1v_0$
$(-2, 0)$	$(1, -1)$	1	1	$(-1, -1)$	$\begin{bmatrix} -u_0v_0 & u_0v_1 \end{bmatrix}$
$(-2, 0)$	$(1, -1)$	1	2	$(0, -2)$	$u_0(dv_1^2 - v_0^2)$
$(-2, 0)$	$(1, 1)$	1	1	$(-1, 1)$	$\begin{bmatrix} u_0v_1 & u_0v_0 \end{bmatrix}$
$(-2, 0)$	$(1, 1)$	1	2	$(0, 2)$	$u_0(-dv_0^2 + v_1^2)$
$(-1, -1)$	$(0, 2)$	1	1	$(-1, 1)$	$\begin{bmatrix} u_0v_0 & u_1v_0 \end{bmatrix}$
$(-1, -1)$	$(0, 2)$	2	1	$(-2, 0)$	$v_0(du_1^2 - u_0^2)$
$(-1, -1)$	$(2, 0)$	1	1	$(1, -1)$	$\begin{bmatrix} u_0v_0 & -u_1v_0 \end{bmatrix}$
$(-1, -1)$	$(2, 0)$	2	1	$(0, -2)$	$v_0(du_1^2 - u_0^2)$
$(-1, 1)$	$(0, -2)$	1	1	$(-1, -1)$	$\begin{bmatrix} u_0v_0 & u_1v_0 \end{bmatrix}$
$(-1, 1)$	$(0, -2)$	2	1	$(-2, 0)$	$v_0(-du_1^2 + u_0^2)$
$(-1, 1)$	$(2, 0)$	1	1	$(1, 1)$	$\begin{bmatrix} -u_1v_0 & -u_0v_0 \end{bmatrix}$
$(-1, 1)$	$(2, 0)$	2	1	$(0, 2)$	$v_0(-du_1^2 + u_0^2)$

α	β	i	j	$i\alpha + j\beta$	$N_{ij}^{\alpha\beta}(u, v)$
(0, -2)	(-1, 1)	1	1	(-1, -1)	$-u_0v_0 \quad -u_0v_1$
(0, -2)	(-1, 1)	1	2	(-2, 0)	$u_0 (dv_1^2 - v_0^2)$
(0, -2)	(1, 1)	1	1	(1, -1)	$u_0v_1 \quad -u_0v_0$
(0, -2)	(1, 1)	1	2	(2, 0)	$u_0 (-dv_0^2 + v_1^2)$
(0, 2)	(-1, -1)	1	1	(-1, 1)	$-u_0v_0 \quad -u_0v_1$
(0, 2)	(-1, -1)	1	2	(-2, 0)	$u_0 (-dv_1^2 + v_0^2)$
(0, 2)	(1, -1)	1	1	(1, 1)	$-u_0v_1 \quad u_0v_0$
(0, 2)	(1, -1)	1	2	(2, 0)	$u_0 (dv_1^2 - v_0^2)$
(1, -1)	(-2, 0)	1	1	(-1, -1)	$u_0v_0 \quad -u_1v_0$
(1, -1)	(-2, 0)	2	1	(0, -2)	$v_0 (-du_1^2 + v_0^2)$
(1, -1)	(0, 2)	1	1	(1, 1)	$u_1v_0 \quad -u_0v_0$
(1, -1)	(0, 2)	2	1	(2, 0)	$v_0 (-du_1^2 + u_0^2)$
(1, 1)	(-2, 0)	1	1	(-1, 1)	$-u_1v_0 \quad -u_0v_0$
(1, 1)	(-2, 0)	2	1	(0, 2)	$v_0 (du_0^2 - u_1^2)$
(1, 1)	(0, -2)	1	1	(1, -1)	$-u_1v_0 \quad u_0v_0$
(1, 1)	(0, -2)	2	1	(2, 0)	$v_0 (du_0^2 - u_1^2)$
(2, 0)	(-1, -1)	1	1	(1, -1)	$-u_0v_0 \quad u_0v_1$
(2, 0)	(-1, -1)	1	2	(0, -2)	$u_0 (-dv_1^2 + v_0^2)$
(2, 0)	(-1, 1)	1	1	(1, 1)	$u_0v_1 \quad u_0v_0$
(2, 0)	(-1, 1)	1	2	(0, 2)	$u_0 (dv_1^2 - v_0^2)$
(-1, -1)	(0, 1)	1	1	(-1, 0)	$-u_0v_3 - u_1v_2 \quad -du_1v_0 - u_0v_1 \quad -du_1v_3 - u_0v_2 \quad -u_0v_0 - u_1v_1$
(-1, -1)	(0, 1)	1	2	(-1, 1)	$\frac{du_1(c_0dv_3^2 - c_0v_2^2 + c_1dv_0^2 - c_1v_1^2)}{2} \quad \frac{u_0(c_0dv_3^2 - c_0v_2^2 + c_1dv_0^2 - c_1v_1^2)}{2}$
(-1, -1)	(0, 1)	2	2	(-2, 0)	0
(-1, -1)	(1, 0)	1	1	(0, -1)	$du_1v_1 - u_0v_0 \quad du_1v_3 - u_0v_2 \quad -u_0v_1 + u_1v_0 \quad -u_0v_3 + u_1v_2$
(-1, -1)	(1, 0)	1	2	(1, -1)	$\frac{du_1(-c_0dv_1^2 + c_0v_0^2 - c_1dv_3^2 + c_1v_2^2)}{2} \quad \frac{u_0(c_0dv_1^2 - c_0v_0^2 + c_1dv_3^2 - c_1v_2^2)}{2}$
(-1, -1)	(1, 0)	2	2	(0, -2)	0
(-1, 0)	(1, -1)	1	1	(0, -1)	$du_0v_1 + u_2v_0 \quad du_3v_1 + u_1v_0 \quad u_0v_0 + u_2v_1 \quad u_1v_1 + u_3v_0$
(-1, 0)	(1, -1)	2	1	(-1, -1)	$\frac{dv_1(c_0du_0^2 - c_0u_2^2 + c_1du_3^2 - c_1u_1^2)}{2} \quad \frac{v_0(-c_0du_0^2 + c_0u_2^2 - c_1du_3^2 + c_1u_1^2)}{2}$
(-1, 0)	(1, -1)	2	2	(0, -2)	0
(-1, 0)	(1, 1)	1	1	(0, 1)	$-u_1v_0 + u_3v_1 \quad -du_3v_0 + u_1v_1 \quad -du_0v_0 + u_2v_1 \quad u_0v_1 - u_2v_0$
(-1, 0)	(1, 1)	2	1	(-1, 1)	$\frac{dv_0(c_0du_0^2 - c_0u_2^2 + c_1du_3^2 - c_1u_1^2)}{2} \quad \frac{v_1(c_0du_0^2 - c_0u_2^2 + c_1du_3^2 - c_1u_1^2)}{2}$
(-1, 0)	(1, 1)	2	2	(0, 2)	0
(-1, 1)	(0, -1)	1	1	(-1, 0)	$-u_0v_2 - u_1v_0 \quad -du_1v_3 - u_0v_1 \quad -du_1v_2 - u_0v_0 \quad -u_0v_3 - u_1v_1$
(-1, 1)	(0, -1)	1	2	(-1, -1)	$\frac{du_1(-c_0dv_2^2 + c_0v_0^2 - c_1dv_3^2 + c_1v_1^2)}{2} \quad \frac{u_0(-c_0dv_2^2 + c_0v_0^2 - c_1dv_3^2 + c_1v_1^2)}{2}$
(-1, 1)	(0, -1)	2	2	(-2, 0)	0
(-1, 1)	(1, 0)	1	1	(0, 1)	$u_0v_3 - u_1v_2 \quad -du_1v_3 + u_0v_2 \quad -du_1v_1 + u_0v_0 \quad u_0v_1 - u_1v_0$
(-1, 1)	(1, 0)	1	2	(1, 1)	$\frac{u_0(c_0dv_1^2 - c_0v_0^2 + c_1dv_3^2 - c_1v_2^2)}{2} \quad \frac{du_1(c_0dv_1^2 - c_0v_0^2 + c_1dv_3^2 - c_1v_2^2)}{2}$
(-1, 1)	(1, 0)	2	2	(0, 2)	0

α	β	i	j	$i\alpha + j\beta$	$N_{ij}^{\alpha\beta}(u, v)$
(0, -1)	(-1, 1)	1	1	(-1, 0)	$\begin{bmatrix} u_0v_1 + u_2v_0 & du_3v_1 + u_1v_0 & du_2v_1 + u_0v_0 & u_1v_1 + u_3v_0 \end{bmatrix}$
(0, -1)	(-1, 1)	2	1	(-1, -1)	$\begin{bmatrix} \frac{dv_1(c_0du_2^2 - c_0u_0^2 + c_1du_3^2 - c_1u_1^2)}{2} & \frac{v_0(c_0du_2^2 - c_0u_0^2 + c_1du_3^2 - c_1u_1^2)}{2} \end{bmatrix}$
(0, -1)	(-1, 1)	2	2	(-2, 0)	0
(0, -1)	(1, 1)	1	1	(1, 0)	$\begin{bmatrix} du_2v_0 + u_0v_1 & u_0v_0 + u_2v_1 & du_3v_0 + u_1v_1 & u_1v_0 + u_3v_1 \end{bmatrix}$
(0, -1)	(1, 1)	2	1	(1, -1)	$\begin{bmatrix} \frac{dv_0(-c_0du_2^2 + c_0u_0^2 - c_1du_3^2 + c_1u_1^2)}{2} & \frac{v_1(c_0du_2^2 - c_0u_0^2 + c_1du_3^2 - c_1u_1^2)}{2} \end{bmatrix}$
(0, -1)	(1, 1)	2	2	(2, 0)	0
(0, 1)	(-1, -1)	1	1	(-1, 0)	$\begin{bmatrix} u_2v_1 + u_3v_0 & du_0v_1 + u_1v_0 & du_3v_1 + u_2v_0 & u_0v_0 + u_1v_1 \end{bmatrix}$
(0, 1)	(-1, -1)	2	1	(-1, 1)	$\begin{bmatrix} \frac{dv_1(-c_0du_3^2 + c_0u_2^2 - c_1du_0^2 + c_1u_1^2)}{2} & \frac{v_0(-c_0du_3^2 + c_0u_2^2 - c_1du_0^2 + c_1u_1^2)}{2} \end{bmatrix}$
(0, 1)	(-1, -1)	2	2	(-2, 0)	0
(0, 1)	(1, -1)	1	1	(1, 0)	$\begin{bmatrix} du_3v_1 - u_2v_0 & u_2v_1 - u_3v_0 & du_0v_1 - u_1v_0 & -u_0v_0 + u_1v_1 \end{bmatrix}$
(0, 1)	(1, -1)	2	1	(1, 1)	$\begin{bmatrix} \frac{v_0(c_0du_3^2 - c_0u_2^2 + c_1du_0^2 - c_1u_1^2)}{2} & \frac{dv_1(-c_0du_3^2 + c_0u_2^2 - c_1du_0^2 + c_1u_1^2)}{2} \end{bmatrix}$
(0, 1)	(1, -1)	2	2	(2, 0)	0
(1, -1)	(-1, 0)	1	1	(0, -1)	$\begin{bmatrix} -du_1v_0 - u_0v_2 & -du_1v_3 - u_0v_1 & -u_0v_0 - u_1v_2 & -u_0v_3 - u_1v_1 \end{bmatrix}$
(1, -1)	(-1, 0)	1	2	(-1, -1)	$\begin{bmatrix} \frac{du_1(-c_0dv_0^2 + c_0v_2^2 - c_1dv_3^2 + c_1v_1^2)}{2} & \frac{u_0(c_0dv_0^2 - c_0v_2^2 + c_1dv_3^2 - c_1v_1^2)}{2} \end{bmatrix}$
(1, -1)	(-1, 0)	2	2	(0, -2)	0
(1, -1)	(0, 1)	1	1	(1, 0)	$\begin{bmatrix} -du_1v_3 + u_0v_2 & u_0v_3 - u_1v_2 & -du_1v_0 + u_0v_1 & u_0v_0 - u_1v_1 \end{bmatrix}$
(1, -1)	(0, 1)	1	2	(1, 1)	$\begin{bmatrix} \frac{u_0(-c_0dv_3^2 + c_0v_2^2 - c_1dv_0^2 + c_1v_1^2)}{2} & \frac{du_1(c_0dv_3^2 - c_0v_2^2 + c_1dv_0^2 - c_1v_1^2)}{2} \end{bmatrix}$
(1, -1)	(0, 1)	2	2	(2, 0)	0
(1, 0)	(-1, -1)	1	1	(0, -1)	$\begin{bmatrix} -du_1v_1 + u_0v_0 & -du_3v_1 + u_2v_0 & -u_0v_1 + u_1v_0 & -u_2v_1 + u_3v_0 \end{bmatrix}$
(1, 0)	(-1, -1)	2	1	(1, -1)	$\begin{bmatrix} \frac{dv_1(c_0du_1^2 - c_0u_0^2 + c_1du_3^2 - c_1u_2^2)}{2} & \frac{v_0(-c_0du_1^2 + c_0u_0^2 - c_1du_3^2 + c_1u_2^2)}{2} \end{bmatrix}$
(1, 0)	(-1, -1)	2	2	(0, -2)	0
(1, 0)	(-1, 1)	1	1	(0, 1)	$\begin{bmatrix} u_2v_1 - u_3v_0 & du_3v_1 - u_2v_0 & du_1v_1 - u_0v_0 & u_0v_1 - u_1v_0 \end{bmatrix}$
(1, 0)	(-1, 1)	2	1	(1, 1)	$\begin{bmatrix} \frac{v_0(-c_0du_1^2 + c_0u_0^2 - c_1du_3^2 + c_1u_2^2)}{2} & \frac{dv_1(-c_0du_1^2 + c_0u_0^2 - c_1du_3^2 + c_1u_2^2)}{2} \end{bmatrix}$
(1, 0)	(-1, 1)	2	2	(0, 2)	0
(1, 1)	(-1, 0)	1	1	(0, 1)	$\begin{bmatrix} u_0v_1 - u_1v_3 & du_0v_3 - u_1v_1 & du_0v_0 - u_1v_2 & u_0v_2 - u_1v_0 \end{bmatrix}$
(1, 1)	(-1, 0)	1	2	(-1, 1)	$\begin{bmatrix} \frac{du_0(-c_0dv_0^2 + c_0v_2^2 - c_1dv_3^2 + c_1v_1^2)}{2} & \frac{u_1(-c_0dv_0^2 + c_0v_2^2 - c_1dv_3^2 + c_1v_1^2)}{2} \end{bmatrix}$
(1, 1)	(-1, 0)	2	2	(0, 2)	0
(1, 1)	(0, -1)	1	1	(1, 0)	$\begin{bmatrix} -du_0v_2 - u_1v_0 & -u_0v_0 - u_1v_2 & -du_0v_3 - u_1v_1 & -u_0v_1 - u_1v_3 \end{bmatrix}$
(1, 1)	(0, -1)	1	2	(1, -1)	$\begin{bmatrix} \frac{du_0(c_0dv_2^2 - c_0v_0^2 + c_1dv_3^2 - c_1v_1^2)}{2} & \frac{u_1(-c_0dv_2^2 + c_0v_0^2 - c_1dv_3^2 + c_1v_1^2)}{2} \end{bmatrix}$
(1, 1)	(0, -1)	2	2	(2, 0)	0

5.1 Commutator identities

$$\alpha = (-2, 0, 0, 0, 0, 0), \quad \beta = (1, -1, 0, 0, 0, 0)$$

$$\left[X_\alpha(u_0), X_\beta \left(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right) \right] = X_{\alpha+\beta} \left(\begin{bmatrix} -u_0v_0 \\ u_0v_1 \end{bmatrix} \right) X_{\alpha+2\beta}(u_0(dv_1^2 - v_0^2))$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -u_0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{d}v_1 + v_0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{d}v_1 - v_0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ u_0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\sqrt{d}v_1 - v_0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{d}v_1 + v_0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -u_0(\sqrt{d}v_1 + v_0) & 1 & 0 & 0 & 0 \\ u_0(\sqrt{d}v_1 - v_0) & u_0(-dv_1^2 + v_0^2) & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[X_\alpha(u_0), X_\beta \left(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right) \right] = X_{\alpha+\beta} \left(\begin{bmatrix} u_0 v_1 \\ u_0 v_0 \end{bmatrix} \right) X_{\alpha+2\beta} (u_0 (-dv_0^2 + v_1^2))$$

$$\left[X_\alpha \left(\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right), X_\beta \left(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right) \right] = X_{\alpha+\beta} (2du_1v_1 - 2u_0v_0)$$

$$\left[X_\alpha \left(\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right), X_\beta \left(\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) \right] = X_{\alpha+\beta} \left(\begin{bmatrix} -u_0 v_3 - u_1 v_2 \\ -du_1 v_0 - u_0 v_1 \\ -du_1 v_3 - u_0 v_2 \\ -u_0 v_0 - u_1 v_1 \end{bmatrix} \right) X_{\alpha+2\beta} \left(\begin{bmatrix} \frac{du_1 (c_0 dv_3^2 - c_0 v_2^2 + c_1 dv_0^2 - c_1 v_1^2)}{2} \\ \frac{u_0 (c_0 dv_3^2 - c_0 v_2^2 + c_1 dv_0^2 - c_1 v_1^2)}{2} \end{bmatrix} \right) X_{2\alpha+2\beta} (0)$$

$$\left[X_\alpha\left(\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}\right), X_\beta(v_0)\right] = X_{\alpha+\beta}\left(\begin{bmatrix} u_0 v_0 \\ u_1 v_0 \end{bmatrix}\right) X_{2\alpha+\beta}(v_0(du_1^2 - u_0^2))$$

$$\left[X_\alpha\left(\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}\right), X_\beta\left(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix}\right)\right] = X_{\alpha+\beta}(-2du_1v_1 - 2u_0v_0)$$

$$\left[X_\alpha \left(\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right), X_\beta \left(\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) \right] = X_{\alpha+\beta} \left(\begin{bmatrix} du_1 v_1 - u_0 v_0 \\ du_1 v_3 - u_0 v_2 \\ -u_0 v_1 + u_1 v_0 \\ -u_0 v_3 + u_1 v_2 \end{bmatrix} \right) X_{\alpha+2\beta} \left(\begin{bmatrix} \frac{du_1(-c_0 dv_1^2 + c_0 v_0^2 - c_1 dv_3^2 + c_1 v_2^2)}{2} \\ \frac{u_0(c_0 dv_1^2 - c_0 v_0^2 + c_1 dv_3^2 - c_1 v_2^2)}{2} \end{bmatrix} \right) X_{2\alpha+2\beta} (0)$$

$$\left[X_\alpha\left(\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}\right), X_\beta(v_0)\right] = X_{\alpha+\beta}\left(\begin{bmatrix} u_0 v_0 \\ -u_1 v_0 \end{bmatrix}\right) X_{2\alpha+\beta}(v_0 (du_1^2 - u_0^2))$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

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[illegible]

$$\left[X_\alpha\left(\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}\right), X_\beta(v_0)\right] = X_{\alpha+\beta}\left(\begin{bmatrix} u_1 v_0 \\ -u_0 v_0 \end{bmatrix}\right) X_{2\alpha+\beta}(v_0(-du_1^2 + u_0^2))$$

$$\left[X_\alpha \left(\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right), X_\beta \left(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right) \right] = X_{\alpha+\beta} (-2du_1v_0 - 2u_0v_1)$$

$$\alpha = (1, 0, 0, 0, 0, 0), \quad \beta = (-1, -1, 0, 0, 0, 0)$$

[illegible]

$$\left[X_\alpha \left(\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} \right), X_\beta \left(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right) \right] = X_{\alpha+\beta} \left(\begin{bmatrix} u_2 v_1 - u_3 v_0 \\ du_3 v_1 - u_2 v_0 \\ du_1 v_1 - u_0 v_0 \\ u_0 v_1 - u_1 v_0 \end{bmatrix} \right) X_{2\alpha+\beta} \left(\begin{bmatrix} \frac{v_0(-c_0 du_1^2 + c_0 u_0^2 - c_1 du_3^2 + c_1 u_2^2)}{2} \\ \frac{dv_1(-c_0 du_1^2 + c_0 u_0^2 - c_1 du_3^2 + c_1 u_2^2)}{2} \end{bmatrix} \right) X_{2\alpha+2\beta}(0)$$

[illegible]

$$\left[X_\alpha \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}, X_\beta \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \right] = X_{\alpha+\beta} \left(\begin{bmatrix} d(c_0 u_0 v_2 - c_0 u_1 v_0 + c_1 u_2 v_3 - c_1 u_3 v_1) \\ -c_0 d u_1 v_2 + c_0 u_0 v_0 - c_1 d u_3 v_3 + c_1 u_2 v_1 \end{bmatrix} \right)$$

[illegible]

$$\left[X_\alpha \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}, X_\beta \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \right] = X_{\alpha+\beta} \left(\begin{bmatrix} c_0 du_1 v_3 - c_0 u_0 v_2 + c_1 du_3 v_0 - c_1 u_2 v_1 \\ d(c_0 u_0 v_3 - c_0 u_1 v_2 + c_1 u_2 v_0 - c_1 u_3 v_1) \end{bmatrix} \right)$$

$$\left[X_{\alpha}(u_0), X_{\beta} \left(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right) \right] = X_{\alpha+\beta} \left(\begin{bmatrix} -u_0 v_0 \\ u_0 v_1 \end{bmatrix} \right) X_{\alpha+2\beta} (u_0 (-dv_1^2 + v_0^2))$$

$$\begin{bmatrix} 1 & 0 & -u_0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & u_0(\sqrt{d}v_1 - v_0) & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & u_0(dv_1^2 - v_0^2) & u_0(\sqrt{d}v_1 + v_0) & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = (2, 0, 0, 0, 0, 0), \quad \beta = (-1, 1, 0, 0, 0, 0)$$

$$\left[X_{\alpha}(u_0), X_{\beta} \left(\begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right) \right] = X_{\alpha+\beta} \left(\begin{bmatrix} u_0 v_1 \\ u_0 v_0 \end{bmatrix} \right) X_{\alpha+2\beta} (u_0 (dv_1^2 - v_0^2))$$

$$\begin{bmatrix} 1 & 0 & -u_0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{d}v_1 + v_0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{d}v_1 - v_0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & u_0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{d}v_1 - v_0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{d}v_1 + v_0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & u_0(-\sqrt{d}v_1 + v_0) & 0 & 0 \\ 0 & 1 & u_0(\sqrt{d}v_1 + v_0) & u_0(-dv_1^2 + v_0^2) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

6 Weyl group

One of the defining characteristics of a root system Φ is that for any two roots $\alpha, \beta \in \Phi$, the reflection of β across the hyperplane perpendicular to α is another root. The reflection across the hyperplane perpendicular to α is denoted σ_α , and the previous sentence can be rephrased as saying that σ_α is a (bijective) function from Φ to itself. The reflection of β is then denoted $\sigma_\alpha(\beta)$. For each root $\alpha \in \Phi$, there is an element $w_\alpha \in G$ with the following properties:

1. $w_\alpha \in N_G(T)$, i.e. w_α normalizes the torus, i.e. $w_\alpha \cdot t \cdot w_\alpha^{-1} \in T$ for all $t \in T$.
2. $w_\alpha^2 \in Z_G(T)$, i.e. w_α^2 centralizes the torus, i.e. $w_\alpha^2 \cdot t \cdot (w_\alpha^2)^{-1} = t$ for all $t \in T$. Under some assumptions, $T = Z_G(T)$, though this is not true for all of the groups considered in this package.
3. For any root $\beta \in \Phi$, the conjugation of the root subgroup U_β by w_α is the root subgroup $U_{\sigma_\alpha(\beta)}$. That is, for $x \in U_\beta$,

$$w_\alpha \cdot x \cdot w_\alpha^{-1} \in U_{\sigma_\alpha(\beta)}$$

More precisely, given an element $x_\beta(u) \in U_\beta$, there is some function $\varphi_{\alpha\beta}(u)$ so that

$$w_\alpha \cdot x_\beta(u) \cdot w_\alpha^{-1} = x_{\sigma_\alpha(\beta)}(\varphi_{\alpha\beta}(u))$$

Taken together, properties (1) and (2) above say that viewed as an element of the quotient group $N_G(T)/Z_G(T)$, (the class of) w_α is an element of order 2 (or order 1 but that doesn't happen for other reasons).

6.1 Weyl group elements

The table below lists roots along with an associated Weyl group element.

α	w_α
$(-2, 0)$	$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
$(-1, -1)$	$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
$(-1, 0)$	$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
$(-1, 1)$	$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

[illegible]

6.2 Weyl group conjugation coefficients

The table below lists roots along with the coefficients obtain in performing conjugation by an associated Weyl element.

α	β	$\gamma = \sigma_\alpha(\beta)$	$\varphi_{\alpha\beta}(u)$
$(-2, 0)$	$(-2, 0)$	$(2, 0)$	$-u_0$
$(-2, 0)$	$(-1, -1)$	$(1, -1)$	$[u_0 \ -u_1]$
$(-2, 0)$	$(-1, 0)$	$(1, 0)$	$[-u_2 \ -u_0 \ -u_1 \ -u_3]$
$(-2, 0)$	$(-1, 1)$	$(1, 1)$	$[-u_1 \ -u_0]$
$(-2, 0)$	$(0, -2)$	$(0, -2)$	u_0
$(-2, 0)$	$(0, -1)$	$(0, -1)$	$[u_0 \ u_1 \ u_2 \ u_3]$
$(-2, 0)$	$(0, 1)$	$(0, 1)$	$[u_0 \ u_1 \ u_2 \ u_3]$
$(-2, 0)$	$(0, 2)$	$(0, 2)$	u_0
$(-2, 0)$	$(1, -1)$	$(-1, -1)$	$[-u_0 \ u_1]$
$(-2, 0)$	$(1, 0)$	$(-1, 0)$	$[u_1 \ u_2 \ u_0 \ u_3]$
$(-2, 0)$	$(1, 1)$	$(-1, 1)$	$[u_1 \ u_0]$
$(-2, 0)$	$(2, 0)$	$(-2, 0)$	$-u_0$
$(-1, -1)$	$(-2, 0)$	$(0, 2)$	$-u_0$
$(-1, -1)$	$(-1, -1)$	$(1, 1)$	$[u_1 \ -u_0]$
$(-1, -1)$	$(-1, 0)$	$(0, 1)$	$[-u_3 \ -u_1 \ -u_2 \ -u_0]$
$(-1, -1)$	$(-1, 1)$	$(-1, 1)$	$[-u_0 \ u_1]$
$(-1, -1)$	$(0, -2)$	$(2, 0)$	$-u_0$
$(-1, -1)$	$(0, -1)$	$(1, 0)$	$[-u_0 \ -u_2 \ -u_1 \ -u_3]$
$(-1, -1)$	$(0, 1)$	$(-1, 0)$	$[u_3 \ u_1 \ u_2 \ u_0]$
$(-1, -1)$	$(0, 2)$	$(-2, 0)$	$-u_0$
$(-1, -1)$	$(1, -1)$	$(1, -1)$	$[-u_0 \ u_1]$
$(-1, -1)$	$(1, 0)$	$(0, -1)$	$[u_0 \ u_2 \ u_1 \ u_3]$
$(-1, -1)$	$(1, 1)$	$(-1, -1)$	$[-u_1 \ u_0]$
$(-1, -1)$	$(2, 0)$	$(0, -2)$	$-u_0$
$(-1, 0)$	$(-2, 0)$	$(2, 0)$	$-u_0$
$(-1, 0)$	$(-1, -1)$	$(1, -1)$	$[u_0 \ -u_1]$
$(-1, 0)$	$(-1, 0)$	$(1, 0)$	$[-u_2 \ -u_0 \ -u_1 \ -u_3]$
$(-1, 0)$	$(-1, 1)$	$(1, 1)$	$[-u_1 \ -u_0]$
$(-1, 0)$	$(0, -2)$	$(0, -2)$	u_0
$(-1, 0)$	$(0, -1)$	$(0, -1)$	$[u_0 \ u_1 \ u_2 \ u_3]$
$(-1, 0)$	$(0, 1)$	$(0, 1)$	$[u_0 \ u_1 \ u_2 \ u_3]$
$(-1, 0)$	$(0, 2)$	$(0, 2)$	u_0
$(-1, 0)$	$(1, -1)$	$(-1, -1)$	$[-u_0 \ u_1]$
$(-1, 0)$	$(1, 0)$	$(-1, 0)$	$[u_1 \ u_2 \ u_0 \ u_3]$
$(-1, 0)$	$(1, 1)$	$(-1, 1)$	$[u_1 \ u_0]$
$(-1, 0)$	$(2, 0)$	$(-2, 0)$	$-u_0$
$(-1, 1)$	$(-2, 0)$	$(0, -2)$	u_0
$(-1, 1)$	$(-1, -1)$	$(-1, -1)$	$[u_0 \ -u_1]$
$(-1, 1)$	$(-1, 0)$	$(0, -1)$	$[u_2 \ u_1 \ u_0 \ u_3]$
$(-1, 1)$	$(-1, 1)$	$(1, -1)$	$[u_0 \ u_1]$
$(-1, 1)$	$(0, -2)$	$(-2, 0)$	u_0
$(-1, 1)$	$(0, -1)$	$(-1, 0)$	$[u_2 \ u_1 \ u_0 \ u_3]$
$(-1, 1)$	$(0, 1)$	$(1, 0)$	$[u_2 \ u_3 \ u_1 \ u_0]$
$(-1, 1)$	$(0, 2)$	$(2, 0)$	u_0
$(-1, 1)$	$(1, -1)$	$(-1, 1)$	$[u_0 \ u_1]$
$(-1, 1)$	$(1, 0)$	$(0, 1)$	$[u_3 \ u_2 \ u_0 \ u_1]$

α	β	$\gamma = \sigma_\alpha(\beta)$	$\varphi_{\alpha\beta}(u)$
(-1, 1)	(1, 1)	(1, 1)	$[-u_0 \ u_1]$
(-1, 1)	(2, 0)	(0, 2)	u_0
(0, -2)	(-2, 0)	(-2, 0)	u_0
(0, -2)	(-1, -1)	(-1, 1)	$[u_0 \ u_1]$
(0, -2)	(-1, 0)	(-1, 0)	$[u_0 \ u_1 \ u_2 \ u_3]$
(0, -2)	(-1, 1)	(-1, -1)	$[-u_0 \ -u_1]$
(0, -2)	(0, -2)	(0, 2)	$-u_0$
(0, -2)	(0, -1)	(0, 1)	$[-u_3 \ -u_1 \ -u_0 \ -u_2]$
(0, -2)	(0, 1)	(0, -1)	$[u_2 \ u_1 \ u_3 \ u_0]$
(0, -2)	(0, 2)	(0, -2)	$-u_0$
(0, -2)	(1, -1)	(1, 1)	$[u_1 \ -u_0]$
(0, -2)	(1, 0)	(1, 0)	$[u_0 \ u_1 \ u_2 \ u_3]$
(0, -2)	(1, 1)	(1, -1)	$[u_1 \ -u_0]$
(0, -2)	(2, 0)	(2, 0)	u_0
(0, -1)	(-2, 0)	(-2, 0)	u_0
(0, -1)	(-1, -1)	(-1, 1)	$[u_0 \ u_1]$
(0, -1)	(-1, 0)	(-1, 0)	$[u_0 \ u_1 \ u_2 \ u_3]$
(0, -1)	(-1, 1)	(-1, -1)	$[-u_0 \ -u_1]$
(0, -1)	(0, -2)	(0, 2)	$-u_0$
(0, -1)	(0, -1)	(0, 1)	$[-u_3 \ -u_1 \ -u_0 \ -u_2]$
(0, -1)	(0, 1)	(0, -1)	$[u_2 \ u_1 \ u_3 \ u_0]$
(0, -1)	(0, 2)	(0, -2)	$-u_0$
(0, -1)	(1, -1)	(1, 1)	$[u_1 \ -u_0]$
(0, -1)	(1, 0)	(1, 0)	$[u_0 \ u_1 \ u_2 \ u_3]$
(0, -1)	(1, 1)	(1, -1)	$[u_1 \ -u_0]$
(0, -1)	(2, 0)	(2, 0)	u_0
(0, 1)	(-2, 0)	(-2, 0)	u_0
(0, 1)	(-1, -1)	(-1, 1)	$[u_0 \ u_1]$
(0, 1)	(-1, 0)	(-1, 0)	$[u_0 \ u_1 \ u_2 \ u_3]$
(0, 1)	(-1, 1)	(-1, -1)	$[-u_0 \ -u_1]$
(0, 1)	(0, -2)	(0, 2)	$-u_0$
(0, 1)	(0, -1)	(0, 1)	$[-u_3 \ -u_1 \ -u_0 \ -u_2]$
(0, 1)	(0, 1)	(0, -1)	$[u_2 \ u_1 \ u_3 \ u_0]$
(0, 1)	(0, 2)	(0, -2)	$-u_0$
(0, 1)	(1, -1)	(1, 1)	$[u_1 \ -u_0]$
(0, 1)	(1, 0)	(1, 0)	$[u_0 \ u_1 \ u_2 \ u_3]$
(0, 1)	(1, 1)	(1, -1)	$[u_1 \ -u_0]$
(0, 1)	(2, 0)	(2, 0)	u_0
(0, 2)	(-2, 0)	(-2, 0)	u_0
(0, 2)	(-1, -1)	(-1, 1)	$[u_0 \ u_1]$
(0, 2)	(-1, 0)	(-1, 0)	$[u_0 \ u_1 \ u_2 \ u_3]$
(0, 2)	(-1, 1)	(-1, -1)	$[-u_0 \ -u_1]$
(0, 2)	(0, -2)	(0, 2)	$-u_0$
(0, 2)	(0, -1)	(0, 1)	$[-u_3 \ -u_1 \ -u_0 \ -u_2]$
(0, 2)	(0, 1)	(0, -1)	$[u_2 \ u_1 \ u_3 \ u_0]$
(0, 2)	(0, 2)	(0, -2)	$-u_0$
(0, 2)	(1, -1)	(1, 1)	$[u_1 \ -u_0]$
(0, 2)	(1, 0)	(1, 0)	$[u_0 \ u_1 \ u_2 \ u_3]$
(0, 2)	(1, 1)	(1, -1)	$[u_1 \ -u_0]$
(0, 2)	(2, 0)	(2, 0)	u_0

α	β	$\gamma = \sigma_\alpha(\beta)$	$\varphi_{\alpha\beta}(u)$
(1, -1)	(-2, 0)	(0, -2)	u_0
(1, -1)	(-1, -1)	(-1, -1)	$[u_0 \ -u_1]$
(1, -1)	(-1, 0)	(0, -1)	$[u_2 \ u_1 \ u_0 \ u_3]$
(1, -1)	(-1, 1)	(1, -1)	$[u_0 \ u_1]$
(1, -1)	(0, -2)	(-2, 0)	u_0
(1, -1)	(0, -1)	(-1, 0)	$[u_2 \ u_1 \ u_0 \ u_3]$
(1, -1)	(0, 1)	(1, 0)	$[u_2 \ u_3 \ u_1 \ u_0]$
(1, -1)	(0, 2)	(2, 0)	u_0
(1, -1)	(1, -1)	(-1, 1)	$[u_0 \ u_1]$
(1, -1)	(1, 0)	(0, 1)	$[u_3 \ u_2 \ u_0 \ u_1]$
(1, -1)	(1, 1)	(1, 1)	$[-u_0 \ u_1]$
(1, -1)	(2, 0)	(0, 2)	u_0
(1, 0)	(-2, 0)	(2, 0)	$-u_0$
(1, 0)	(-1, -1)	(1, -1)	$[u_0 \ -u_1]$
(1, 0)	(-1, 0)	(1, 0)	$[-u_2 \ -u_0 \ -u_1 \ -u_3]$
(1, 0)	(-1, 1)	(1, 1)	$[-u_1 \ -u_0]$
(1, 0)	(0, -2)	(0, -2)	u_0
(1, 0)	(0, -1)	(0, -1)	$[u_0 \ u_1 \ u_2 \ u_3]$
(1, 0)	(0, 1)	(0, 1)	$[u_0 \ u_1 \ u_2 \ u_3]$
(1, 0)	(0, 2)	(0, 2)	u_0
(1, 0)	(1, -1)	(-1, -1)	$[-u_0 \ u_1]$
(1, 0)	(1, 0)	(-1, 0)	$[u_1 \ u_2 \ u_0 \ u_3]$
(1, 0)	(1, 1)	(-1, 1)	$[u_1 \ u_0]$
(1, 0)	(2, 0)	(-2, 0)	$-u_0$
(1, 1)	(-2, 0)	(0, 2)	$-u_0$
(1, 1)	(-1, -1)	(1, 1)	$[u_1 \ -u_0]$
(1, 1)	(-1, 0)	(0, 1)	$[-u_3 \ -u_1 \ -u_2 \ -u_0]$
(1, 1)	(-1, 1)	(-1, 1)	$[-u_0 \ u_1]$
(1, 1)	(0, -2)	(2, 0)	$-u_0$
(1, 1)	(0, -1)	(1, 0)	$[-u_0 \ -u_2 \ -u_1 \ -u_3]$
(1, 1)	(0, 1)	(-1, 0)	$[u_3 \ u_1 \ u_2 \ u_0]$
(1, 1)	(0, 2)	(-2, 0)	$-u_0$
(1, 1)	(1, -1)	(1, -1)	$[-u_0 \ u_1]$
(1, 1)	(1, 0)	(0, -1)	$[u_0 \ u_2 \ u_1 \ u_3]$
(1, 1)	(1, 1)	(-1, -1)	$[-u_1 \ u_0]$
(1, 1)	(2, 0)	(0, -2)	$-u_0$
(2, 0)	(-2, 0)	(2, 0)	$-u_0$
(2, 0)	(-1, -1)	(1, -1)	$[u_0 \ -u_1]$
(2, 0)	(-1, 0)	(1, 0)	$[-u_2 \ -u_0 \ -u_1 \ -u_3]$
(2, 0)	(-1, 1)	(1, 1)	$[-u_1 \ -u_0]$
(2, 0)	(0, -2)	(0, -2)	u_0
(2, 0)	(0, -1)	(0, -1)	$[u_0 \ u_1 \ u_2 \ u_3]$
(2, 0)	(0, 1)	(0, 1)	$[u_0 \ u_1 \ u_2 \ u_3]$
(2, 0)	(0, 2)	(0, 2)	u_0
(2, 0)	(1, -1)	(-1, -1)	$[-u_0 \ u_1]$
(2, 0)	(1, 0)	(-1, 0)	$[u_1 \ u_2 \ u_0 \ u_3]$
(2, 0)	(1, 1)	(-1, 1)	$[u_1 \ u_0]$
(2, 0)	(2, 0)	(-2, 0)	$-u_0$

$$\alpha = (-1, -1, 0, 0, 0, 0), \beta = (1, 1, 0, 0, 0, 0), \sigma_\alpha(\beta) = (-1, -1, 0, 0, 0, 0)$$

$$\alpha = (-1, -1, 0, 0, 0, 0), \beta = (2, 0, 0, 0, 0, 0), \sigma_\alpha(\beta) = (0, -2, 0, 0, 0, 0)$$

$$\alpha = (-1, 1, 0, 0, 0, 0), \beta = (-2, 0, 0, 0, 0, 0), \sigma_\alpha(\beta) = (0, -2, 0, 0, 0, 0)$$

$$\alpha = (-1, 1, 0, 0, 0, 0), \beta = (-1, -1, 0, 0, 0, 0), \sigma_{\alpha}(\beta) = (-1, -1, 0, 0, 0, 0)$$

$$\alpha = (-1, 1, 0, 0, 0, 0), \beta = (-1, 0, 0, 0, 0, 0), \sigma_\alpha(\beta) = (0, -1, 0, 0, 0, 0)$$

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