An algebraic approach to musical scales

Joshua Ruiter

GAUSS seminar

October 15, 2020

MICHIGAN STATE UNIVERSITY

Outline

1. What is a scale mathematically?
   - Frequency and intervals
   - Octave equivalence
   - Overtones
   - Transposition
   - Subgroups of S^1
2. Comparing equal temperaments
   - Rational approximation
   - Continued fractions
   - The “best" equal temperament

Definition

A **pitch** or **frequency** or **note** is a positive real number with units of $\frac{1}{\text{sec}}$. We'll mostly forget about the units, so the set of pitches is $\mathbb{R}_{>0}$. Bigger number $\iff$ higher pitch.

Definition

A **pitch** or **frequency** or **note** is a positive real number with units of $\frac{1}{\text{sec}}$. We'll mostly forget about the units, so the set of pitches is $\mathbb{R}_{>0}$. Bigger number $\iff$ higher pitch.

Example

Orchestras tune to the note 440 Hz. Sample

Definition

A **pitch** or **frequency** or **note** is a positive real number with units of $\frac{1}{\text{sec}}$. We'll mostly forget about the units, so the set of pitches is $\mathbb{R}_{>0}$. Bigger number $\iff$ higher pitch.

Example

Orchestras tune to the note 440 Hz. Sample

Definition

An **interval** or **frequency ratio** is a ratio of two frequencies. This is the "distance" between them. As a set, intervals are also $\mathbb{R}_{>0}$. Bigger number $\iff$ bigger jump in pitch.

Definition

A **pitch** or **frequency** or **note** is a positive real number with units of $\frac{1}{\text{sec}}$. We'll mostly forget about the units, so the set of pitches is $\mathbb{R}_{>0}$. Bigger number $\iff$ higher pitch.

Example

Orchestras tune to the note 440 Hz. Sample

Definition

An **interval** or **frequency ratio** is a ratio of two frequencies. This is the "distance" between them. As a set, intervals are also $\mathbb{R}_{>0}$. Bigger number $\iff$ bigger jump in pitch.

Example

A **perfect fourth** is the interval $\frac{4}{3}$. Sample *Here comes the bride*
A **perfect fifth** is the interval $\frac{3}{2}$. Sample *Twinkle, twinkle, little star*
An **octave** is the interval 2. Sample *Somewhere over the rainbow*

Definition

A **scale** is a set of intervals containing 1. $S \subset \mathbb{R}_{>0}, 1 \in S$.

Definition

A **scale** is a set of intervals containing 1. $S \subset \mathbb{R}_{>0}, 1 \in S$.

Definition

Fix a scale S. A piece of music P is **in tune** with respect to S if for any two notes $x, y \in P$, the intervals $\frac{x}{y}$ and $\frac{y}{x}$ are in S.

Definition

A **scale** is a set of intervals containing 1. $S \subset \mathbb{R}_{>0}, 1 \in S$.

Definition

Fix a scale S. A piece of music P is **in tune** with respect to S if for any two notes $x, y \in P$, the intervals $\frac{x}{y}$ and $\frac{y}{x}$ are in S.

Example

The set $S = \left\{1, \frac{3}{2}\right\}$ is a very simple scale. Using this scale, a piece of music is only in tune if it contains only perfect fifths.

Definition

A **scale** is a set of intervals containing 1. $S \subset \mathbb{R}_{>0}, 1 \in S$.

Definition

Fix a scale S. A piece of music P is **in tune** with respect to S if for any two notes $x, y \in P$, the intervals $\frac{x}{y}$ and $\frac{y}{x}$ are in S.

Example

The set $S = \left\{1, \frac{3}{2}\right\}$ is a very simple scale. Using this scale, a piece of music is only in tune if it contains only perfect fifths.

Example

The set $S = \mathbb{R}_{>0}$ is a scale. With respect to this scale, everything is in tune.

Example

The **Pythagorean scale** is the set

$$S = \left\{1, \frac{3}{2}, \left(\frac{3}{2}\right)^2, \ldots, \left(\frac{3}{2}\right)^{11}\right\}$$

It has 12 intervals.

Example

The **Pythagorean scale** is the set

$$S = \left\{1, \frac{3}{2}, \left(\frac{3}{2}\right)^2, \dots, \left(\frac{3}{2}\right)^{11}\right\}$$

It has 12 intervals.

Example

Let $n \in \mathbb{Z}, n \geq 1$. The ***n*-tone equal tempered scale** (nTET) is the set

$$S = \left\{1, 2^{1/n}, 2^{2/n}, 2^{3/n}, \dots\right\}$$

At the moment this technically has infinitely many intervals, but later we'll say this only has n distinct intervals. 12TET, 19TET

Definition

Two intervals are **octave equivalent** if they differ by some number of octaves. Mathematically, they differ (multiplicatively) by an integer power of 2.

From now on, we consider octave equivalent intervals to be the same interval.

Definition

Two intervals are **octave equivalent** if they differ by some number of octaves. Mathematically, they differ (multiplicatively) by an integer power of 2.

From now on, we consider octave equivalent intervals to be the same interval. Mathematically, we are taking the group $(\mathbb{R}_{>0}, \times)$ of intervals and quotienting out by the integer powers of 2.

$$G = \mathbb{R}_{>0} / \{2^n : n \in \mathbb{Z}\}$$

From now on, an interval is identified with its coset in this quotient group. Given a coset, we can always find a unique representative in the interval $[1, 2)$, so we often conflate a coset with its representative in this interval. From now on, a scale is a set of interval cosets under octave equivalence.

Example

After quotienting by 2, we can rescale the intervals in the Pythagorean scale so that everything is between 1 and 2.

$$S = \left\{1, \frac{3}{2}, \left(\frac{3}{2}\right)^2, \ldots, \left(\frac{3}{2}\right)^{11}\right\} = \left\{1, \frac{3}{2}, \frac{9}{8}, \frac{27}{16} \ldots, \frac{3^{11}}{2^5}\right\}$$

Example

After quotienting by 2, we can rescale the intervals in the Pythagorean scale so that everything is between 1 and 2.

$$S = \left\{1, \frac{3}{2}, \left(\frac{3}{2}\right)^2, \ldots, \left(\frac{3}{2}\right)^{11}\right\} = \left\{1, \frac{3}{2}, \frac{9}{8}, \frac{27}{16} \ldots, \frac{3^{11}}{2^5}\right\}$$

Example

Recall the nTET scale.

$$S = \left\{1, 2^{1/n}, 2^{2/n}, 2^{3/n}, \ldots\right\}$$

In the quotient G, we can write any $2^{m/n}$ as $2^{m'/n}$ where $0 \leq m' \leq n-1$. For example, $2^{(n+1)/n} = 2^{n/n}2^{1/n} = 2^1 2^{1/n} = 2^{1/n}$ by octave equivalence. So nTET actually has only n distinct intervals, up to octave equivalence.

The functions e^x and log are group isomorphisms

$$(\mathbb{R}, +) \underset{\log}{\overset{\exp}{\rightleftarrows}} (\mathbb{R}_{>0}, \times)$$

The functions e^x and log are group isomorphisms

$$(\mathbb{R}, +) \underset{\log}{\overset{\exp}{\rightleftarrows}} (\mathbb{R}_{>0}, \times)$$

So the quotients are also isomorphic.

$$(\mathbb{R}/\log 2, +) \underset{\log}{\overset{\exp}{\rightleftarrows}} (\mathbb{R}_{>0}/2, \times)$$

With some scaling, $\mathbb{R}/\log 2 \cong \mathbb{R}/\mathbb{Z}$.

The functions e^x and log are group isomorphisms

$$(\mathbb{R}, +) \underset{\log}{\overset{\exp}{\rightleftarrows}} (\mathbb{R}_{>0}, \times)$$

So the quotients are also isomorphic.

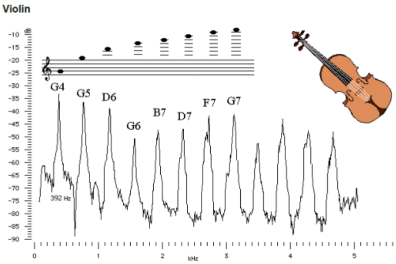
$$(\mathbb{R}/\log 2, +) \underset{\log}{\overset{\exp}{\rightleftarrows}} (\mathbb{R}_{>0}/2, \times)$$

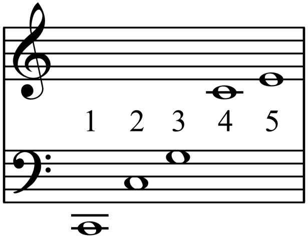
With some scaling, $\mathbb{R}/\log 2 \cong \mathbb{R}/\mathbb{Z}$. As yet another way to think about it, $\mathbb{R}/\mathbb{Z}$ is isomorphic to the unit circle in the complex plane.

$$\mathbb{R}/\mathbb{Z} \xrightarrow{\cong} S^1 \qquad x \mapsto e^{2\pi i x}$$

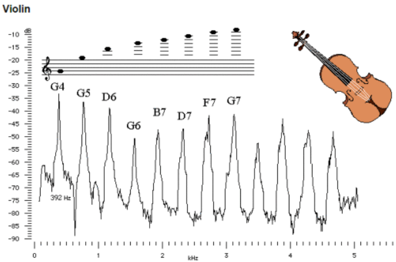
So mathematically, a scale is a subset of S^1.

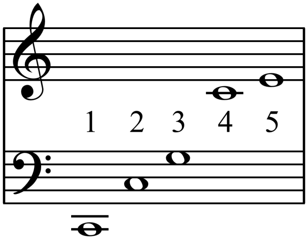
When a frequency x is played on a string or wind instrument, the instrument also produces less prominent frequencies at positive integer multiples of x. The prominence of multiples decreases as they get higher.
Overtones





When a frequency x is played on a string or wind instrument, the instrument also produces less prominent frequencies at positive integer multiples of x. The prominence of multiples decreases as they get higher. Overtones





Definition

In the situation above, x is the **fundamental frequency** and the frequencies $2x, 3x, 4x, \ldots$ are **overtones**.

Example

Suppose we simultaneously play x and $\frac{3}{2}x$ as fundamental frequencies on two different violins. The resulting sounding overtones are

$$x : 2x, 3x, 4x, 5x, 6x, \dots \qquad \frac{3}{2}x : 3x, \frac{9}{2}x, 6x, \dots$$

These two fundamental frequencies share a lot of overtones: $3x, 6x, 9x, \dots$. These common overtones are a big part of why the interval $\frac{3}{2}$ sounds "good." $\frac{3}{2}$ interval

Example

Suppose we simultaneously play x and $\frac{3}{2}x$ as fundamental frequencies on two different violins. The resulting sounding overtones are

$$x : 2x, 3x, 4x, 5x, 6x, \ldots \qquad \frac{3}{2}x : 3x, \frac{9}{2}x, 6x, \ldots$$

These two fundamental frequencies share a lot of overtones: $3x, 6x, 9x, \ldots$. These common overtones are a big part of why the interval $\frac{3}{2}$ sounds “good." $\frac{3}{2}$ interval

Example

In contrast, suppose we simultaneously play x and $2^{7/12}x$ as fundamental frequencies. Now none of the overtones are equal, because an integer multiple of $2^{7/12}$ can’t be rational. But they’re close.

$$x : 2x, 3x, \ldots \qquad 2^{7/12}x : 2 \cdot 2^{7/12}x \approx 2.9966x, \ldots$$

$2^{7/12}$ interval

Definition

Let A be a set of frequencies and let x be an interval. The **transposition** of P by x is the set $\{xp : p \in P\}$.

Example

Transposition sample

Definition

Let A be a set of frequencies and let x be an interval. The **transposition** of P by x is the set $\{xp : p \in P\}$.

Example

Transposition sample

Definition

A scale S is **transposition invariant** if it remains the same under any transposition by an interval $x \in S$. That is, for any $x \in S$, $xS = S$.

Example

The nTET scale is transposition invariant. $S = \left\{2^{m/n} : m \in \mathbb{Z}\right\}$ is clearly closed under multiplication.

Example

The nTET scale is transposition invariant. $S = \left\{2^{m/n} : m \in \mathbb{Z}\right\}$ is clearly closed under multiplication.

Example

The Pythagorean scale is not transposition invariant. $S = \left\{\left(\frac{3}{2}\right)^n : n = 0, \dots 11\right\}$. If we take $x = \frac{3}{2}$ and multiply it by the last note $\left(\frac{3}{2}\right)^{11}$ we get $\left(\frac{3}{2}\right)^{12} \approx 129.7$ which is not in the scale. Note however that $129.7 \approx 128 = 2^7$, so the new note is closer to being in the scale than it looks, since $1 = 128$ by octave equivalence. If we rescale to between 1 and 2, the new note is

$$\left(\frac{3}{2}\right)^{12} 2^{-7} \approx 1.0136\dots$$

which is quite close to 1. This very small interval is called the **Pythagorean comma**. Sample

Lemma

A scale is transposition invariant if and only if it is a subgroup of S^1.

Lemma

A scale is transposition invariant if and only if it is a subgroup of S^1.

Proof.

Clearly being a subgroup implies transposition invariance.

Lemma

A scale is transposition invariant if and only if it is a subgroup of S^1.

Proof.

Clearly being a subgroup implies transposition invariance. Suppose S is transposition invariant. By definition $1 \in S$, so it just suffices to show S is closed under inverses.

Lemma

A scale is transposition invariant if and only if it is a subgroup of S^1.

Proof.

Clearly being a subgroup implies transposition invariance. Suppose S is transposition invariant. By definition $1 \in S$, so it just suffices to show S is closed under inverses. Let $x \in S$. Since $S = xS$, there exists $y \in S$ so that $xy = 1$, so $y = x^{-1} \in S$. $\square$

Lemma

A scale is transposition invariant if and only if it is a subgroup of S^1.

Proof.

Clearly being a subgroup implies transposition invariance. Suppose S is transposition invariant. By definition $1 \in S$, so it just suffices to show S is closed under inverses. Let $x \in S$. Since $S = xS$, there exists $y \in S$ so that $xy = 1$, so $y = x^{-1} \in S$. □

Remark

Recall $G = \mathbb{R}_{>0}/2$.

1. The finite order elements of G are of the form $2^{m/n}$ where $n, m \in \mathbb{Z}$.

Lemma

A scale is transposition invariant if and only if it is a subgroup of S^1.

Proof.

Clearly being a subgroup implies transposition invariance. Suppose S is transposition invariant. By definition $1 \in S$, so it just suffices to show S is closed under inverses. Let $x \in S$. Since $S = xS$, there exists $y \in S$ so that $xy = 1$, so $y = x^{-1} \in S$. $\square$

Remark

Recall $G = \mathbb{R}_{>0}/2$.

1. The finite order elements of G are of the form $2^{m/n}$ where $n, m \in \mathbb{Z}$.
2. G has a unique cyclic subgroup of every finite order n, generated by $2^{1/n}$. More generally, any $2^{m/n}$ satisfying $\gcd(m, n) = 1$ is a generator. (This subgroup is the nTET scale.)

Lemma

A scale is transposition invariant if and only if it is a subgroup of S^1.

Proof.

Clearly being a subgroup implies transposition invariance. Suppose S is transposition invariant. By definition $1 \in S$, so it just suffices to show S is closed under inverses. Let $x \in S$. Since $S = xS$, there exists $y \in S$ so that $xy = 1$, so $y = x^{-1} \in S$. □

Remark

Recall $G = \mathbb{R}_{>0}/2$.

1. The finite order elements of G are of the form $2^{m/n}$ where $n, m \in \mathbb{Z}$.
2. G has a unique cyclic subgroup of every finite order n, generated by $2^{1/n}$. More generally, any $2^{m/n}$ satisfying $\gcd(m, n) = 1$ is a generator. (This subgroup is the nTET scale.)
3. If $\gcd(a, b) = 1$, then $\gcd(a + b, ab) = 1$.

Lemma

Any finite subgroup of G is cyclic.

Lemma

Any finite subgroup of G is cyclic.

Proof.

Let $H \subset G$ be a finite subgroup. By classification finite abelian groups,

$$H \cong \mathbb{Z}/a_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/a_i\mathbb{Z}$$

where $a_1, \ldots, a_i$ are pairwise relatively prime.

Lemma

Any finite subgroup of G is cyclic.

Proof.

Let $H \subset G$ be a finite subgroup. By classification finite abelian groups,

$$H \cong \mathbb{Z}/a_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/a_i\mathbb{Z}$$

where $a_1, \ldots, a_i$ are pairwise relatively prime. By induction it suffices to prove H is cyclic (of order relatively prime to $a_3, \ldots, a_i$) in the case $i = 2$. So assume $H \cong (\mathbb{Z}/a\mathbb{Z}) \oplus (\mathbb{Z}/b\mathbb{Z})$, $\gcd(a, b) = 1$.

Lemma

Any finite subgroup of G is cyclic.

Proof.

Let $H \subset G$ be a finite subgroup. By classification finite abelian groups,

$$H \cong \mathbb{Z}/a_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/a_i\mathbb{Z}$$

where $a_1, \ldots, a_i$ are pairwise relatively prime. By induction it suffices to prove H is cyclic (of order relatively prime to $a_3, \ldots, a_i$) in the case $i = 2$. So assume $H \cong (\mathbb{Z}/a\mathbb{Z}) \oplus (\mathbb{Z}/b\mathbb{Z})$, $\gcd(a, b) = 1$. By remark 2, these two cyclic subgroups are generated by $2^{1/a}$ and $2^{1/b}$ respectively.

Lemma

Any finite subgroup of G is cyclic.

Proof.

Let $H \subset G$ be a finite subgroup. By classification finite abelian groups,

$$H \cong \mathbb{Z}/a_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/a_i\mathbb{Z}$$

where $a_1, \ldots, a_i$ are pairwise relatively prime. By induction it suffices to prove H is cyclic (of order relatively prime to $a_3, \ldots, a_i$) in the case $i = 2$. So assume $H \cong (\mathbb{Z}/a\mathbb{Z}) \oplus (\mathbb{Z}/b\mathbb{Z})$, $\gcd(a, b) = 1$. By remark 2, these two cyclic subgroups are generated by $2^{1/a}$ and $2^{1/b}$ respectively. Then H contains the element

$$x = 2^{1/a}2^{1/b} = 2^{1/a+1/b} = 2^{\frac{a+b}{ab}}$$

Lemma

Any finite subgroup of G is cyclic.

Proof.

Let $H \subset G$ be a finite subgroup. By classification finite abelian groups,

$$H \cong \mathbb{Z}/a_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/a_i\mathbb{Z}$$

where $a_1, \ldots, a_i$ are pairwise relatively prime. By induction it suffices to prove H is cyclic (of order relatively prime to $a_3, \ldots, a_i$) in the case $i = 2$. So assume $H \cong (\mathbb{Z}/a\mathbb{Z}) \oplus (\mathbb{Z}/b\mathbb{Z})$, $\gcd(a, b) = 1$. By remark 2, these two cyclic subgroups are generated by $2^{1/a}$ and $2^{1/b}$ respectively. Then H contains the element

$$x = 2^{1/a}2^{1/b} = 2^{1/a+1/b} = 2^{\frac{a+b}{ab}}$$

By remark 3, $\gcd(a + b, ab) = 1$, so by remark 2 x generates a cyclic subgroup of H of order ab, which is the order of H, so $H = \langle x \rangle$. $\square$

Corollary

If a scale is transposition invariant and finite, it is an equal tempered scale.

Corollary

If a scale is transposition invariant and finite, it is an equal tempered scale.

Remark

Equal tempered scales never contain rational intervals. So no nTET scale includes a perfect fifth $\frac{3}{2}$, pefect fourth $\frac{4}{3}$, etc.

Corollary

If a scale is transposition invariant and finite, it is an equal tempered scale.

Remark

Equal tempered scales never contain rational intervals. So no nTET scale includes a perfect fifth $\frac{3}{2}$, pefect fourth $\frac{4}{3}$, etc.

Next question: are some nTET scales better than others? How to decide?

Corollary

If a scale is transposition invariant and finite, it is an equal tempered scale.

Remark

Equal tempered scales never contain rational intervals. So no nTET scale includes a perfect fifth $\frac{3}{2}$, pefect fourth $\frac{4}{3}$, etc.

Next question: are some nTET scales better than others? How to decide? Let's say we want to choose an nTET scale with a good approximation for a perfect fifth.

n	nTET approximation to $\frac{3}{2}$
2	$2^{1/2} \approx 1.414$
3	$2^{2/3} \approx 1.587$
5	$2^{3/5} \approx 1.516$
12	$2^{7/12} \approx 1.498$
19	$2^{11/19} \approx 1.494$

Outline

Choosing n so that nTET has a good approximation for a perfect fifth is equivalent to asking

For which n is there an m so that $2^{m/n}$ is a good approximation for $\frac{3}{2}$?

Choosing n so that nTET has a good approximation for a perfect fifth is equivalent to asking

> For which n is there an m so that $2^{m/n}$ is a good approximation for $\frac{3}{2}$?

Let's reframe the question slightly, so that instead of approximation a rational with irrationals, we approximation an irrational with rationals.

$$2^{m/n} \approx \frac{3}{2} \iff \frac{m}{n} \approx \log_2 \frac{3}{2} = \log_2 3 - \log_2 2 = \log_2 3 - 1 \iff \frac{m+n}{n} \approx \log_2 3$$

Choosing n so that nTET has a good approximation for a perfect fifth is equivalent to asking

> For which n is there an m so that $2^{m/n}$ is a good approximation for $\frac{3}{2}$?

Let's reframe the question slightly, so that instead of approximation a rational with irrationals, we approximation an irrational with rationals.

$$2^{m/n} \approx \frac{3}{2} \iff \frac{m}{n} \approx \log_2 \frac{3}{2} = \log_2 3 - \log_2 2 = \log_2 3 - 1 \iff \frac{m+n}{n} \approx \log_2 3$$

So the question has become

> What are good rational approximations for $\log_2 3$?

Definition

Let $x \in \mathbb{R}$. A (reduced) rational number $\frac{c}{d}$ is a **best approximation of the first kind** for x if

$$\left|x - \frac{c}{d}\right| < \left|x - \frac{p}{q}\right|$$

for any fraction $\frac{p}{q}$ with $q < d$. In other words, $\left|x - \frac{c}{d}\right|$ is minimal among such expressions involving rational numbers with denominator bounded by d. That is, $\frac{c}{d}$ is the best rational approximation of x except for fractions with larger denominator.

Definition

Let $x \in \mathbb{R}$. A (reduced) rational number $\frac{c}{d}$ is a **best approximation of the first kind** for x if

$$\left|x - \frac{c}{d}\right| < \left|x - \frac{p}{q}\right|$$

for any fraction $\frac{p}{q}$ with $q < d$. In other words, $\left|x - \frac{c}{d}\right|$ is minimal among such expressions involving rational numbers with denominator bounded by d. That is, $\frac{c}{d}$ is the best rational approximation of x except for fractions with larger denominator.

Definition

Let $x \in \mathbb{R}$. A (reduced) rational number $\frac{c}{d}$ is a **best approximation of the second kind** for x if

$$|dx - c| < |qx - p|$$

for any fraction $\frac{p}{q}$ with $q < d$.

Definition

A **continued fraction** is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

We notate it more succinctly as

$$[a_0; a_1, a_2, a_3, \ldots]$$

Definition

A **continued fraction** is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

We notate it more succinctly as

$$[a_0; a_1, a_2, a_3, \ldots]$$

Theorem

Every real number x can be written as a continued fraction. If x is irrational, then the continued fraction representation is unique.

Algorithm (to compute continued fraction of x)

1. *Set* $x_0 = x$, *and set* $a_0 = \lfloor x_0 \rfloor = \lfloor x \rfloor$.
2. *Set* $x_{i+1} = \frac{1}{x_i - a_i}$, *and set* $a_{i+1} = \lfloor x_{i+1} \rfloor$. *(If at any point* $x_i = a_i$ *the algorithm terminates because* x *is rational.)*

Algorithm (to compute continued fraction of x)

1. *Set* $x_0 = x$, *and set* $a_0 = \lfloor x_0 \rfloor = \lfloor x \rfloor$.
2. *Set* $x_{i+1} = \frac{1}{x_i - a_i}$, *and set* $a_{i+1} = \lfloor x_{i+1} \rfloor$. *(If at any point* $x_i = a_i$ *the algorithm terminates because* x *is rational.)*

Example (Continued fraction for $\log_2 3$)

$$x_0 = \log_2 3 \approx 1.58\ldots$$

Algorithm (to compute continued fraction of x)

1. *Set* $x_0 = x$*, and set* $a_0 = \lfloor x_0 \rfloor = \lfloor x \rfloor$.
2. *Set* $x_{i+1} = \frac{1}{x_i - a_i}$*, and set* $a_{i+1} = \lfloor x_{i+1} \rfloor$. *(If at any point* $x_i = a_i$ *the algorithm terminates because* x *is rational.)*

Example (Continued fraction for $\log_2 3$)

$$x_0 = \log_2 3 \approx 1.58\ldots \qquad a_0 = \lfloor x_0 \rfloor = 1$$

Algorithm (to compute continued fraction of x)

1. *Set $x_0 = x$, and set $a_0 = \lfloor x_0 \rfloor = \lfloor x \rfloor$.*
2. *Set $x_{i+1} = \frac{1}{x_i - a_i}$, and set $a_{i+1} = \lfloor x_{i+1} \rfloor$. (If at any point $x_i = a_i$ the algorithm terminates because x is rational.)*

Example (Continued fraction for $\log_2 3$)

$$x_0 = \log_2 3 \approx 1.58\ldots \qquad a_0 = \lfloor x_0 \rfloor = 1$$

$$x_1 = \frac{1}{x_0 - a_0} = \frac{1}{\log_2 3 - 1} \approx 1.70\ldots$$

Algorithm (to compute continued fraction of x)

1. *Set* $x_0 = x$, *and set* $a_0 = \lfloor x_0 \rfloor = \lfloor x \rfloor$.
2. *Set* $x_{i+1} = \frac{1}{x_i - a_i}$, *and set* $a_{i+1} = \lfloor x_{i+1} \rfloor$. *(If at any point* $x_i = a_i$ *the algorithm terminates because* x *is rational.)*

Example (Continued fraction for $\log_2 3$)

$$x_0 = \log_2 3 \approx 1.58\ldots \qquad a_0 = \lfloor x_0 \rfloor = 1$$

$$x_1 = \frac{1}{x_0 - a_0} = \frac{1}{\log_2 3 - 1} \approx 1.70\ldots \qquad a_1 = \lfloor x_1 \rfloor = 1$$

Algorithm (to compute continued fraction of x)

1. *Set* $x_0 = x$, *and set* $a_0 = \lfloor x_0 \rfloor = \lfloor x \rfloor$.
2. *Set* $x_{i+1} = \frac{1}{x_i - a_i}$, *and set* $a_{i+1} = \lfloor x_{i+1} \rfloor$. *(If at any point* $x_i = a_i$ *the algorithm terminates because* x *is rational.)*

Example (Continued fraction for $\log_2 3$)

$$x_0 = \log_2 3 \approx 1.58\ldots \qquad a_0 = \lfloor x_0 \rfloor = 1$$

$$x_1 = \frac{1}{x_0 - a_0} = \frac{1}{\log_2 3 - 1} \approx 1.70\ldots \qquad a_1 = \lfloor x_1 \rfloor = 1$$

$$x_2 = \frac{1}{x_1 - a_1} = \frac{1}{\frac{1}{\log_2 3 - 1} - 1} \approx 1.40\ldots$$

Algorithm (to compute continued fraction of x)

1. *Set $x_0 = x$, and set $a_0 = \lfloor x_0 \rfloor = \lfloor x \rfloor$.*
2. *Set $x_{i+1} = \frac{1}{x_i - a_i}$, and set $a_{i+1} = \lfloor x_{i+1} \rfloor$. (If at any point $x_i = a_i$ the algorithm terminates because x is rational.)*

Example (Continued fraction for $\log_2 3$)

$$x_0 = \log_2 3 \approx 1.58 \ldots \qquad a_0 = \lfloor x_0 \rfloor = 1$$

$$x_1 = \frac{1}{x_0 - a_0} = \frac{1}{\log_2 3 - 1} \approx 1.70 \ldots \qquad a_1 = \lfloor x_1 \rfloor = 1$$

$$x_2 = \frac{1}{x_1 - a_1} = \frac{1}{\frac{1}{\log_2 3 - 1} - 1} \approx 1.40 \ldots \qquad a_2 = \lfloor x_2 \rfloor = 1$$

Algorithm (to compute continued fraction of x)

1. *Set $x_0 = x$, and set $a_0 = \lfloor x_0 \rfloor = \lfloor x \rfloor$.*
2. *Set $x_{i+1} = \frac{1}{x_i - a_i}$, and set $a_{i+1} = \lfloor x_{i+1} \rfloor$. (If at any point $x_i = a_i$ the algorithm terminates because x is rational.)*

Example (Continued fraction for $\log_2 3$)

$$x_0 = \log_2 3 \approx 1.58\ldots \qquad a_0 = \lfloor x_0 \rfloor = 1$$

$$x_1 = \frac{1}{x_0 - a_0} = \frac{1}{\log_2 3 - 1} \approx 1.70\ldots \qquad a_1 = \lfloor x_1 \rfloor = 1$$

$$x_2 = \frac{1}{x_1 - a_1} = \frac{1}{\frac{1}{\log_2 3 - 1} - 1} \approx 1.40\ldots \qquad a_2 = \lfloor x_2 \rfloor = 1$$

$$\log_2 3 = [1; 1, 1, 2, 2, 3, 1, 5, 2, 23, 2, 2, 1, 1, 55, \ldots] \qquad = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2 + \ldots}}}}$$

Definition

Let $x = [a_0; a_1, a_2, \ldots]$. The kth **convergent** of x is the rational number $c_k(x) = [a_0; a_1, \ldots, a_k]$.

Definition

Let $x = [a_0; a_1, a_2, \ldots]$. The kth **convergent** of x is the rational number $c_k(x) = [a_0; a_1, \ldots, a_k]$.

Example (Convergents of $\log_2 3$)

$$c_0 = 1$$

$$c_1 = 1 + \frac{1}{1} = 2$$

$$c_2 = 1 + \frac{1}{1 + \frac{1}{1}} = \frac{3}{2}$$

$$c_3 = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{2}}} = \frac{8}{5}$$

$$c_4 = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{2+\frac{1}{2}}}} = \frac{19}{12}$$

$$c_5 = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{3}}}}} = \frac{65}{41}$$

Theorem

Let $x \in \mathbb{R}, x \notin \mathbb{Z}$. A fraction is a best rational approximation of the second kind for x if and only if it is a convergent of x.

Theorem

Let $x \in \mathbb{R}, x \notin \mathbb{Z}$. A fraction is a best rational approximation of the second kind for x if and only if it is a convergent of x.

So the first few best approximations for $\log_2 3$ are $1, 2, \frac{3}{2}, \frac{8}{5}, \frac{19}{12}, \frac{65}{41}, \ldots$.

Theorem

Let $x \in \mathbb{R}, x \notin \mathbb{Z}$. A fraction is a best rational approximation of the second kind for x if and only if it is a convergent of x.

So the first few best approximations for $\log_2 3$ are $1, 2, \frac{3}{2}, \frac{8}{5}, \frac{19}{12}, \frac{65}{41}, \ldots$. Our approximation problem was

$$2^{m/n} \approx \frac{3}{2} \iff \frac{m+n}{n} \approx \log_2 3$$

So we want our convergent to be $\frac{m+n}{n}$.

Theorem

Let $x \in \mathbb{R}, x \notin \mathbb{Z}$. A fraction is a best rational approximation of the second kind for x if and only if it is a convergent of x.

So the first few best approximations for $\log_2 3$ are $1, 2, \frac{3}{2}, \frac{8}{5}, \frac{19}{12}, \frac{65}{41}, \ldots$. Our approximation problem was

$$2^{m/n} \approx \frac{3}{2} \iff \frac{m+n}{n} \approx \log_2 3$$

So we want our convergent to be $\frac{m+n}{n}$. The table says that 2TET, 5TET, 12TET, 41TET, etc. have good approximations for a perfect fifth.

k	$\frac{n+m}{n}$	n	m	Approximation for $\frac{3}{2}$
2	$3/2$	2	1	$2^{1/2} \approx 1.414$
3	$8/5$	5	3	$2^{3/5} \approx 1.516$
4	$19/12$	12	7	$2^{7/12} \approx 1.4983$
5	$65/41$	41	24	$2^{24/41} \approx 1.50042$

Thank you!

References and places to learn more

Benson, D.J. *Music: a Mathematical Offering*. Cambridge University Press, 2008. https://homepages.abdn.ac.uk/d.j.benson/pages/html/maths-music.html

A. Ya. Khinchin, *Continued fractions*. Translated from the third (1961) Russian edition, Reprint of the 1964 translation, Dover, Mineola, NY, 1997.

Lamb, Evelyn. “What’s So Great about Continued Fractions?” Roots of Unity, Scientific American, 17 Mar. 2015, blogs.scientificamerican.com/roots-of-unity/what-8217-s-so-great-about-continued-fractions/.