Lie algebras and their root systems

A case study in the classification of Lie algebras

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Calvin College Mathematics Colloquium
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Historical background

Source for historical information: Helgason, Sigurdur (1994), "Sophus Lie, the Mathematician," Proceedings of The Sophus Lie Memorial Conference, Oslo, August, 1992, Oslo: Scandinavian University Press, pp. 3–21.

The matrix algebra $\mathrm{sl}(3, \mathbb{C})$

$\mathrm{sl}(3, \mathbb{C})$ is the set of 3×3 matrices with complex entries and trace zero.

$$\mathrm{sl}(3, \mathbb{C}) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} : a + e + i = 0 \right\}$$

This is a vector space, because:

- The zero matrix has trace zero.
- A scalar multiple of a traceless matrix is traceless.
- The sum of two traceless matrices is traceless.

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Basis matrices

Definition

The matrix e_{ij} is a matrix with a one in the ijth place and zeroes elsewhere.

Usual basis for $\mathfrak{sl}(3, \mathbb{C})$:

$$\{e_{11} - e_{22}, e_{22} - e_{33}\} \cup \{e_{ij} : i \neq j\}$$

Direct sum expression for $\mathfrak{sl}(3, \mathbb{C})$:

$$\mathfrak{sl}(3, \mathbb{C}) = \operatorname{span}\{e_{11} - e_{22}, e_{22} - e_{33}\} \oplus \bigoplus_{i \neq j} \operatorname{span}\{e_{ij}\}$$

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Is it a ring?

We just determined that $\mathfrak{sl}(3, \mathbb{C})$ is closed under matrix addition. Is it closed under matrix multiplication?

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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A new binary operation

Definition

Let x, y be matrices. The **bracket** of x and y is $[A, B] := AB - BA$.

Proposition

Let A, B be matrices. Then $[A, B]$ has trace zero.

Proof.

From linear algebra, we know that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and tr is linear. Hence $\operatorname{tr}[A, B] = \operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0$. □

Corollary

$A, B \in \mathrm{sl}(3, \mathbb{C}) \implies [A, B] \in \mathrm{sl}(3, \mathbb{C})$

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Properties of bracket (1)

Proposition (Bracket is alternating)

Let A be a matrix. Then $[A, A] = 0$.

Proof.

$[A, A] = A^2 - A^2 = 0$ □

Proposition (Bracket is antisymmetric)

Let A, B be matrices. Then $[A, B] = -[B, A]$.

Proof.

$[A, B] = AB - BA = -BA + AB = -(BA - AB) = -[B, A]$ □

Corollary

Let $A, B \in \mathfrak{sl}(3, \mathbb{C})$. Then $[A, B] = [B, A] \iff [A, B] = 0$.

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Properties of bracket (2)

Proposition (Bracket is bilinear)

Let A, B, C be matrices and let λ be a scalar. Then

$$\begin{aligned}
[A+B, C] &= [A, C] + [B, C] \\
[C, A+B] &= [C, A] + [C, B] \\
[\lambda A, C] &= [A, \lambda C] = \lambda [A, C]
\end{aligned}$$

Proof.

Apply the fact that matrix multiplication distributes over matrix addition and commutes with scalar multiplication. □

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Properties of bracket (3)

Proposition (Jacobi identity)

Let A, B, C be matrices. Then $[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0$.

Proof.

Remember that matrix multiplication distributes over matrix addition. Expand all the terms out, get a lot of terms like ABC, and see that they all cancel in pairs. $\square$

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Brackets of basis matrices

Definition

The **Kronecker delta** function is the function

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Lemma

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}$$

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Definition

A **Lie algebra** is a vector space L over a field F with a bilinear form $[,] : L \times L \to L$ that satisfies $[x, x] = 0$ and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for $x, y, z \in L$.

As a consequence of this definition, the bracket must be antisymmetric.

Definition

$\mathrm{SL}(3,\mathbb{C}) = \{A \in M(3,\mathbb{C}) : \det A = 1\}$

$$\alpha(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t^2+1 & t^3+2t \\ 0 & t & t^2+1 \end{pmatrix} \qquad \alpha(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(\alpha(t)) = 1 \implies \alpha(t) \in \mathrm{SL}(3,\mathbb{C})$$

$$\alpha'(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2t & 3t^2+2 \\ 0 & 1 & 2t \end{pmatrix} \qquad \alpha'(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

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Brief excursion: connection to Lie groups

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Definition

Let V, W be vector spaces. A **linear map** is a function $\phi : V \to W$ such that $\phi(av_1 + v_2) = a\phi(v_1) + \phi(v_2)$.

For each $x \in \mathfrak{sl}(3, \mathbb{C})$, we can associate to x a map from $L \to \mathbb{C}$ by $y \to [x, y]$. This map associated to x is called $\operatorname{ad} x$.

Proposition

For $x \in \mathfrak{sl}(3, \mathbb{C})$, $\operatorname{ad} x$ is linear.

Proof.

Follows from linearity of the bracket in the 2nd entry. $\square$

Linear maps and ad x

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The subalgebra of diagonal matrices

$$H = \left\{ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} : d_1 + d_2 + d_3 = 0 \right\} \subset \mathrm{sl}(3, \mathbb{C})$$

Proposition

Let $A, B \in H$. Then $[A, B] = 0$, hence $[A, B] \in H$.

Proof.

A, B are diagonal, so $AB = BA$, so $[A, B] = AB - BA = 0$. □

Definition

A **subalgebra** of a Lie algebra L is a vector subspace H that is closed under the bracket ($x, y \in H \implies [x, y] \in H$).

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Ideals: $\mathfrak{sl}(3, \mathbb{C})$ is simple

Definition

An **ideal** of a Lie algebra L is a vector subspace I such that for $x \in L, a \in I, [x, a] \in I$.

Note that every ideal is a subalgebra, but not every subalgebra is an ideal.

Proposition

$\mathfrak{sl}(3, \mathbb{C})$ *is simple, that is, it has no nonzero proper ideals.*

Definition

A Lie algebra L is **semisimple** if it can be written as a direct sum of simple Lie algebras.

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Sketch of proof that $\mathrm{sl}(n, \mathbb{C})$ is simple

Suppose I is a nonzero proper ideal with $v \neq 0, v \in I$. Let

$$v = \sum_{i \neq j} c_{ij} e_{ij} + \sum_{i=1}^{n} d_i e_{ii}$$

where $c_{ij}, d_i \in \mathbb{C}$. Then

$$[e_{kl}, [e_{kl}, v]] = -2c^{lk} e_{kl}$$
$$[e_{kl}, v] = (d^l - d^k) e_{kl}$$

From this it follows that $e_{ij} \in I$ for some $i \neq j$. One can then show that if an ideal of $\mathrm{sl}(n, \mathbb{C})$ contains some e_{ij}, then $I = \mathrm{sl}(n, \mathbb{C})$.

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Let h be the diagonal matrix with entries d_1, d_2, d_3. Then

$$\operatorname{ad} h(e_{ij}) = [h, e_{ij}] = he_{ij} - e_{ij}h = d_i e_{ij} - d_j e_{ij} = (d_i - d_j)e_{ij}$$

Definition

A linear map $\phi : V \to V$ is **diagonalizable** if there is a basis of V consisting of eigenvectors for ϕ.

Specifically, $\operatorname{ad} h$ is diagonalizable.

Diagonal subalgebra, continued

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Simultaneous diagonalization

Proposition (Lemma 16.7)

Let $x_1, \dots x_k : V \to V$ be diagonalizable linear transformations. There is a basis of V the simultaneously diagonalizes all x_i if and only if each pair x_i, x_j commutes.

Application: If $h_1, h_2, \dots h_k$ are diagonal elements of $\mathfrak{sl}(3, \mathbb{C})$, then $\operatorname{ad} h_1, \operatorname{ad} h_2, \dots \operatorname{ad} h_k$ are all simultaneously diagonalized in the usual basis, so they all pairwise commute.

Roots (1)

Let $h = \mathrm{diag}(d_1, d_2, d_3)$, and let $L_{ij} = \mathrm{span}\{e_{ij}\}$. Based on our computations so far, we have

$$H \subset \{x \in \mathrm{sl}(3, \mathbb{C}) : \mathrm{ad}\, h(x) = 0, \text{ for all } h \in H\}$$
$$L_{ij} \subset \{x \in \mathrm{sl}(3, \mathbb{C}) : \mathrm{ad}\, h(x) = (d_i - d_j)x, \text{ for all } h \in H\}$$

Actually, we can replace $\subset$ with $=$, this takes a bit more work. Define $\epsilon_i : H \to \mathbb{C}$ by $\epsilon_i(\mathrm{diag}(d_1, d_2, d_3)) = d_i$. Then we can rewrite this as

$$L_{ij} = \{x \in \mathrm{sl}(3, \mathbb{C}) : \mathrm{ad}\, h(x) = (\epsilon_i - \epsilon_j)(h)x, \text{ for all } h \in H\}$$

This makes L_{ij} a **root space** with associated **root** $(\epsilon_i - \epsilon_j)$.

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Roots (2)

Definition

A **root** is a linear map $\alpha : H \to \mathbb{C}$ such that

$$\{x \in \mathrm{sl}(3, \mathbb{C}) : \mathrm{ad}\, h(x) = \alpha(h)x \text{ for all } h \in H\}$$

is a nonzero subspace of $\mathrm{sl}(3, \mathbb{C})$.

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Proposition

$\Phi = \{\epsilon_i - \epsilon_j : i \neq j\}$ *is the entire set of roots for* $\mathrm{sl}(3, \mathbb{C})$.

Recall:

$$\mathrm{sl}(3, \mathbb{C}) = \mathrm{span}\{e_{11} - e_{22}, e_{22} - e_{33}\} \oplus \bigoplus_{i \neq j} \mathrm{span}\{e_{ij}\}$$

Now that we have the language of roots, we can write this as

$$\mathrm{sl}(3, \mathbb{C}) = H \oplus \bigoplus_{i \neq j} L_{ij} = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

This is called the root space decomposition.

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Root systems

Definition

A **root system** is a subset R of a real inner-product space E satisfying:
(R1) R is finite, it spans E, and it does not contain 0.
(R2) If $\alpha \in R$, then the only scalar multiples of α in R are $\pm\alpha$.
(R3) If $\alpha \in R$, then the reflection s_α permutes R.
(R4) If $\alpha, \beta \in R$, then $2(\alpha, \beta)/(\beta, \beta) \in \mathbb{Z}$.

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Let L be a complex semisimple Lie algebra, and let Φ be a set of roots of L. Then Φ is a root system.

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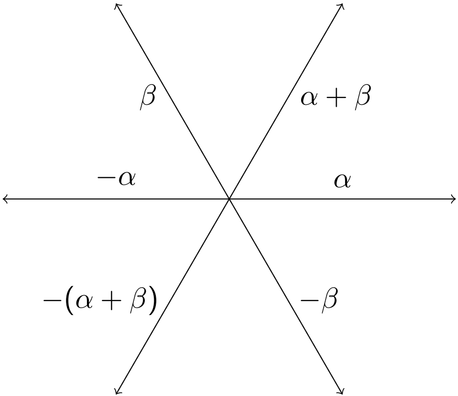
Root diagram for $\mathfrak{sl}(3, \mathbb{C})$



$$\alpha = \epsilon_1 - \epsilon_2 \qquad \beta = \epsilon_2 - \epsilon_3$$

Classification by root systems

Theorem (Cartan)

Up to isomorphism, there is just one complex semisimple Lie algebra for each root system.

Theorem (Cartan)

With five exceptions, every finite-dimensional simple Lie algebra over $\mathbb{C}$ is isomorphic to one of the classical Lie algebras

$$\mathrm{sl}(n, \mathbb{C}) \qquad \mathrm{so}(n, \mathbb{C}) \qquad \mathrm{sp}(2n, \mathbb{C})$$

The five exceptional Lie algebras are known as e_6, e_7, e_8, f_4, and g_2.

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