1 Proof by contradiction

Date: Friday, February 14

1.1 Irrationality of $\sqrt{2}$

Lemma 1. *If n^2 is even, then n is even.*

Proof. We prove the contrapositive. Suppose n is odd. Then $n = 2k+1$ for some integer k. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $2k^2 + 2k$ is an integer, n^2 is odd. □

Proposition 1. *There does not exist a rational number such that $q^2 = 2$.*

Proof. Suppose for the sake of contradiction that there does exist a rational number q so that $q^2 = 2$. Since q is rational, we can write it as a fraction. We don't just want to write it in any way, though. We'll make sure that the fraction is already reduced. So we find integers a, b so that $q = \frac{a}{b}$, and a, b have no common factors greater than 1. Then

$$2 = q^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$$

Ok now we rearrange this into

$$2b^2 = a^2$$

Since b^2 is an integer, we conclude that a^2 is even. Then by the lemma, a is even. So $a = 2m$ for some integer m. Then

$$2b^2 = (2m)^2 = 4m^2 \implies b^2 = 2m^2$$

Since m^2 is an integer, b^2 must be even. So b must be even. Aha! We have a contradiction. We know that a, b have no common factors greater than 1, but we also know they are both even which means they have a common factor of 2. This contradiction leads us to reject our contradiction supposition, namely that q exists. So we conclude that no q exists which makes $q^2 = 2$. □