Fractals and the Koch snowflake

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April 19, 2020

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1 Koch snowflake

1.1 Construction

First, we should start with what the Koch[1] snowflake is. It is a set of points in the plane, but which points? Since I cannot describe them all at once, I will instead describe how to build or construct the Koch snowflake in steps. At step zero, start with an equilateral triangle of side length 1. This is a relatively tame mathematical object - we know understand its properties quite well, namely area and perimeter.

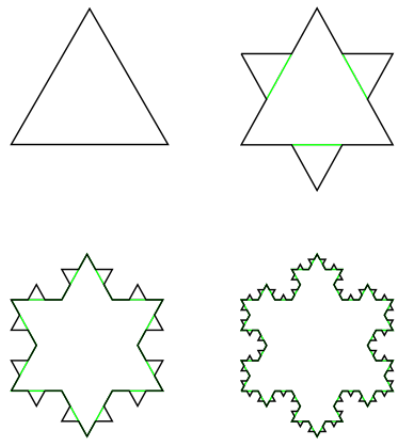
At step 1, divide each side of the triangle into equal thirds. From the middle segment of each side, build a smaller equaliteral triangle with the middle segment as the base, pointing outwards. Then erase the base of those new triangles. The resulting shape is (the boundary of) a six-pointed star.

The whole process in step 1 which we applied to each side is going to be called "one step" or "one iteration." In step 2, we do this process to each side of our new star shape. It has 12 different sides. Each of those 12 sides gets divided into 3 equal segments, and each middle segment is replaced by a littel tent.

In step 3, we do this again, to all of our sides. And so on and so for, doing this pattern infinitely many times. At this point, we're losing some mathematical rigor. Is it really possible to do this infinitely many times? Does that make sense? We're going to skip over

[1] Pronounced like "coke."

those details, since they're quite hard to address from a rigorous mathematical perspective. It can be done, I assure you, but it would take lot more time for us to get into that.



1.2 Counting sides

Let's first determine the number of sides after n steps. Let's give this sequence of side numbers a name, S_n. This will be the number of sides after n steps. So

$$S_0 = 3 \qquad S_1 = 12 \qquad S_2 =?$$

What happens to S_n each time we iterate? Our process involves replacing each side with 4 new sides, so $S_{n+1} = 4S_n$. Ok, so that means that

$$S_n = 3 \cdot 4^n$$

Technicall I should do an induction proof to prove that, but it's not a very hard induction proof, so I'll leave it to you as an exercise.

1.3 Area

Now let's try to compute the area. It's not immediately clear that we can do this. I mean, the thing is very strange. Maybe area doesn't have the same meaning for something like this. Or maybe it's just really hard to compute. It turns out, the area isn't actually that hard to compute. Let's do it in steps.

First, what is the area of the original equaliteral triangle? The base is 1. Using the Pythagorean theorem, we find the the height is $\frac{\sqrt{3}}{2}$, so the area of the triangle is $\frac{\sqrt{3}}{4}$. I'll call this A_0.

$$A_0 = \frac{\sqrt{3}}{4}$$

What happens when we do the first step of the iterative process, and add some more triangles? Those triangles have base $\frac{1}{3}$. Since their side length is $\frac{1}{3}$ of the original triangle, their area is $\frac{1}{9}$ of the area. So each one has area $\frac{\sqrt{3}}{4 \cdot 9}$. How many of those triangles are there? There are 3. So we've added $3 \cdot \frac{\sqrt{3}}{4 \cdot 9}$ units of area to our original. So after 1 iteration, the total area is

$$A_1 = A_0 + 3 \cdot \frac{\sqrt{3}}{4 \cdot 9} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}$$

Instead of trying to do another step "manually," let's back up and think more broadly. What happens each time we do one iteration? First of all, we can observe that the area never decreases. Each time, we just add on more area.

Rather than try to figure out how much area is added each time, let's try to answer an easier question. How many triangles do we add at each step? We add one new triangle for each side. So the number of triangles added at step n is just S_{n-1}, the number of sides from the previous step. So at the nth step, we add

$$S_{n-1} = 3 \cdot 4^{n-1}$$

new triangles. Let's also name the area of each new triangle at step n, we'll call it B_n. Let's say that $B_0 = \frac{\sqrt{3}}{4}$, the area of the original triangle. We already figured out that

$$B_1 = \frac{1}{9} A_0 = \frac{1}{9} B_0$$

Each time we iterate, the new triangle has side length equal to $\frac{1}{3}$ of the previous new triangle, so it has area equal to $\frac{1}{9}$ of the previous new triangle. So in general,

$$B_n = \frac{1}{9} B_{n-1}$$

There's another induction argument we should do here, but the result that we care about is a general formula for B_n.

$$B_n = \frac{B_0}{9^n} = \frac{A_0}{9^n}$$

Now we have all the pieces to figure out the area at the nth step, in terms of the previous area. At the nth step, we add S_{n-1} new triangles, each with area B_n. So

$$A_n = A_{n-1} + B_n S_{n-1} = A_{n-1} + \frac{A_0}{9^n} \cdot 3 \cdot 4^{n-1} = A_{n-1} + \frac{3A_0 4^{n-1}}{9^n}$$

I want to write this without the $n-1$ power, so I'll just multiply by $\frac{4}{4}$.

$$A_n = A_{n-1} + \frac{3A_0 4^n}{4 \cdot 9^n} = A_{n-1} + \frac{3}{4} A_0 \left(\frac{4}{9}\right)^n$$

Now we can start writing out what the areas at each step are.

$$
\begin{aligned}
A_0 &= A_0 \\
A_1 &= A_0 + \frac{3}{4}A_0\left(\frac{4}{9}\right) \\
A_2 &= A_0 + \frac{3}{4}A_0\left(\frac{4}{9}\right) + \frac{3}{4}A_0\left(\frac{4}{9}\right)^2 \\
A_3 &= A_0 + \frac{3}{4}A_0\left(\frac{4}{9}\right) + \frac{3}{4}A_0\left(\frac{4}{9}\right)^2 + \frac{3}{4}A_0\left(\frac{4}{9}\right)^3 \\
&\vdots \\
A_n &= A_0 + \frac{3}{4}A_0\left(\frac{4}{9}\right) + \cdots + \frac{3}{4}A_0\left(\frac{4}{9}\right)^n \\
&= A_0 + \frac{3}{4}A_0\left(\frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^n\right)
\end{aligned}
$$

Let's call $r = \frac{4}{9}$, so we can write this as

$$A_n = A_0 + \frac{3}{4}A_0\left(r + r^2 + \cdots + r^n\right) = A_0 + \frac{3A_0 r}{4}\left(1 + r + r^2 + \cdots r^{n-1}\right)$$

At this point, I'm going to do another bit of non-rigorous math. I'm going to claim that since A_n is the area of the nth step, the area of the Koch snowflake after infinitely many steps should be the limit

$$A = \lim_{n\to\infty} A_n$$

There's a lot of sophisticated math behind that claim, but we don't have time today. In any case, this limit is really just the sum of an infinite series.

$$A = A_0 + \frac{3A_0 r}{4}(1 + r + r^2 + r^3 + \cdots)$$

Does this infinite series make sense? In other words, does it converge? Why yes, yes it does. It's just a geometric series with common ratio $r = \frac{4}{9}$, and since $|r| < 1$, it converges. In particular, it converges to

$$1 + r + r^2 + \cdots = \frac{1}{1-r} = \frac{1}{1-\frac{4}{9}} = \frac{1}{\frac{5}{9}} = \frac{9}{5}$$

So the area of the Koch snowflake is

$$A = A_0 + \frac{3A_0 r}{4}\cdot\frac{9}{5} = \frac{\sqrt{3}}{4} + \frac{3}{4}\cdot\frac{\sqrt{3}}{4}\cdot\frac{4}{9}\cdot\frac{9}{5} = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20} = \frac{5\sqrt{3}}{20} + \frac{3\sqrt{3}}{20} = \frac{8\sqrt{3}}{20} = \frac{2\sqrt{3}}{5}$$

In terms of decimal approximations for comparison,

$$A_0 = \frac{\sqrt{3}}{4} \approx 0.433 \qquad A = \frac{2\sqrt{3}}{5} \approx 0.683$$

To be precise, the Koch snowflake has exactly $\frac{2/5}{1/4} = \frac{8}{5} = 1.6$ times as much area as the original triangle.

1.4 Perimeter

After the calculation ordeal which was the area, you'd expect the perimeter to be just as bad. Lucky for us, the perimeter is way easier, although the resulting answer is quite surprising. The perimeter at the start is

$$P_0 = 3$$

In each step, we replace $\frac{1}{3}$ of each side with two sides of that length. So the new perimeter is $\frac{4}{3}$ as much.

$$P_n = \frac{4}{3} P_{n-1}$$

We could use induction to prove that

$$P_n = \left(\frac{4}{3}\right)^n P_0 = \left(\frac{4}{3}\right)^n \cdot 3$$

After infinitely many steps, we get the Koch snowflake, and so its perimter should be the limit of this sequence.

$$P = \lim_{n\to\infty} P_n = \lim_{n\to\infty} \left(\frac{4}{3}\right)^n \cdot 3$$

What is this limit? Wait a second. This limit is positive infinity. In terms of the formal definition of limits we developed in this class the sequence diverges. What does this mean in terms of the geometry and shapes? Informally speaking, this means that the Koch snowflake has... infinite perimeter?

Actually, yes! The Koch snowflake has infinite perimeter. This is not just an informal way of speaking. It is a fact, in the right mathematical framework and using some complicated definitions and theorems, that we can say that the Koch snowflake has infinite perimeter.

Remark 1.1. To summarize, the Koch snowflake has finite area (we computed it) and infinite perimeter. This is very strange. Normal geometric objects like rectangles, triangles, and circles do not have infinite perimeter. This is a the sort of thing which can happen with fractals, which does not happen with regular geometric objects.

2 Fractals in general

Last time we talked about the area and perimeter of the Koch snowflake. Today my goal is to talk about the fractal dimension of the Koch snowflake. But to get there, first we need to talk about dimension more broadly, then fractal dimension, and only then can we try to figure out the dimension of the Koch snowflake.

2.1 Dimension and scaling factors

Let's start by talking about line segments, squares, and cubes. For each of these, we have a naturally associated positive real number.

Object	Measure
Line segment	Length
Square	Area
Cube	Volume

What I want to spend some time thinking about is what goes on when we scale these objects by some constant. What happens to the associated measurement? Let's start by just thinking about doubling. In the case of the square and cube, I mean double the side length.

Object	Original measurement	Measurement after doubling
Line segment	L	$2L$
Square	A	$4A$
Cube	V	$8V$

Ok, that's fine. Now what about if we do tripling instead?

Object	Original measurement	Measurement after doubling
Line segment	L	$2L$
Square	A	$9A$
Cube	V	$27V$

On the other hand, what if we do multiplying by $\frac{1}{2}$ instead, or $\frac{1}{3}$?

Object	Original	After scaling $\frac{1}{2}$	After scaling $\frac{1}{3}$
Line segment	L	$\frac{1}{2}L$	$\frac{1}{3}L$
Square	A	$\frac{1}{4}A$	$\frac{1}{9}A$
Cube	V	$\frac{1}{8}V$	$\frac{1}{27}V$

Let's back up and think a bit more generally. Let's imagine that we're going to scale everything by a factor of k. We'll scale the line by k, we'll scale the side length of the square by k, and we'll scale the side length of the cube by k. What happens? The length of the line gets scaled by k. The area of the square gets scaled by k^2. The volume of the cube gets scaled by k^3.

Object	Original measurement	Measurement after scaling by k
Line segment	L	kL
Square	A	k^2A
Cube	V	k^3V

Let me include another table with all of the doubling, tripling, halving, and $\frac{1}{3}$-ing which is written in a way which makes it look more like this k-scaling factor generalization.

Object	Original	Scaling by k	Scaling by 2	Scaling by 3	Scaling by $\frac{1}{2}$	Scaling by $\frac{1}{3}$
Line segment	L	k^1L	2^1L	3^1L	$\left(\frac{1}{2}\right)^1 L$	$\left(\frac{1}{3}\right)^1 L$
Square	A	k^2A	2^2L	3^2L	$\left(\frac{1}{2}\right)^2 L$	$\left(\frac{1}{3}\right)^2 L$
Cube	V	k^3V	2^3V	3^3V	$\left(\frac{1}{2}\right)^3 V$	$\left(\frac{1}{3}\right)^3 V$

In each row of this table, there is one constant which sticks out (besides the original measurement). That constant is 1 in the first row, 2 in the second row, and 3 in the third row. It is the exponent which we need to raise our scaling factor k by to get the new measurement. That number is what we'll call the dimension of the shape. Let me try to give a slightly more formal definition of dimension.

Definition 2.1. Let X be a geometric shape with original measurement m (this could be length, area, volume, etc.). Let $k > 0$ be a positive real number (a "scaling factor"). Let $\widetilde{X}$ be the object X scaled by a factor of k, then let $\widetilde{m}$ be the measurement of $\widetilde{X}$. This satisfies the equation

$$\widetilde{m} = k^d m$$

The number d appearing in this equation is the **dimension** of X.

Example 2.2. Let's apply this definition of dimension to lines, squares, and cubes.

1. Scaling a line by a factor of k scales the length of the line by a factor of k^1. So the dimension is 1.

2. Scaling a square (scaling the side length) by a factor of k scales the area by k^2, so the dimension is 2.

3. Scaling a cube (scaling the side length) by a factor of k scales the area by k^3, so the dimension is 3.

Example 2.3. Let's see what the dimension of a triangle (with respect to measuring area) is according to our definition. Intuitively speaking, a triangle ought to be a 2-dimensional object, so if our definition doesn't say that, perhaps we should change our definition.

All right, let's say we have a triangle T with base b and height h. The area is $A = \frac{1}{2}bh$. Let $k > 0$ be a scaling factor. When we scale T by k, we multiply each side by k. Our new triangle $\widetilde{T}$ has base kb and height kh. The area (measure) of $\widetilde{T}$ is

$$\widetilde{A} = \frac{1}{2}(kb)(kh) = k^2\left(\frac{1}{2}bh\right) = k^2 A$$

So our dimension is 2, since we square the scaling constant.

Example 2.4. Let's apply our definition to find the dimension of a sphere (with respect to volume). Intuition says the dimension ought to be 3, I hope it works out. We have a sphere S, with radius r. The volume (measure) is

$$V = \frac{4}{3}\pi r^3$$

Let $k > 0$ be a scaling factor. We scale the sphere by k, which means scaling the radius by k. So the new radius is kr, for our scaled sphere $\widetilde{S}$. The new volume is

$$\widetilde{V} = \frac{4}{3}\pi(kr)^3 = k^3\frac{4}{3}\pi r^3 = k^3 V$$

So our scaling factor gets cubed, so the dimension is 3.

Remark 2.5. One important thing to note when using our definition of dimension is that we don't actually need to do a computation for a general scaling factor k. We can get to the dimension d by using just one scaling factor.

Let's take the example of area of a circle C with radius r. The area is $A = \pi r^2$. We'll choose a scaling factor of $k = 10$. Then we get a new circle $\widetilde{C}$ with radius $10r$, whose area is $\widetilde{A} = \pi(10r)^2 = 10^2\pi r^2 = k^2 A$. We can tell from this calculation that the dimension is 2, even though we didn't consider general scaling factors k.

As a slightly more abstract example, if we have an object X with measurement m, and we want to find the dimension, we can do the following. Pick any scaling factor $k > 0$ which is convenient, scale X by k to obtain a scaled object $\widetilde{X}$, then compute the measurement $\widetilde{m}$ of $\widetilde{X}$. You can then write down the equation

$$\widetilde{m} = k^d m$$

In this equation, you know $\widetilde{m}, k, m$. We could also write this as

$$\frac{\widetilde{m}}{m} = d^k$$

This is useful, since to use this formula, we don't even need to know what m and $\widetilde{m}$ are, we just need to know their ratio. Anyway, we can solve for d, using logarithms. In particular, we solve for d and get

$$d = \log_k\left(\frac{\widetilde{m}}{m}\right) = \log_k \widetilde{m} - \log_k m$$

or in terms of any logarithm base,

$$d = \frac{\log \frac{\widetilde{m}}{m}}{\log k} = \frac{\log \widetilde{m} - \log m}{\log k}$$

Example 2.6. Let's apply the previous kind of logic to talk about the dimension of a square. We already know the dimension is 2, but let's think about it a slightly new way.

Let's pretend for the moment that we don't even know how to compute areas of squares in terms of the side length. Instead of an arbitrary scaling factor k, we make the following observation. When we scale the side length by a factor of $k = \frac{1}{2}$, we get a new square, and we can cover our original square with exactly 4 copies of our original square. All we know is that a scale factor of $\frac{1}{2}$ results in something which is $\frac{1}{4}$ as big. So in this situation

$$\frac{m}{\widetilde{m}} = \frac{1}{4}$$

So we write down the equation involving the dimension d as

$$\frac{1}{4} = k^d = \left(\frac{1}{2}\right)^d$$

We could solve this using logarithms, but it's also pretty obvious that we don't need to, it's clear here that $d = 2$.

2.2 Dimension of the Koch curve

Now let's try to apply our notion of dimension above to the Koch snowflake and its area. First, we need to decide what it means to scale a Koch snowflake. This isn't actually too tricky, but it's also not terribly obvious. To scale up a Koch snowflake by a scaling factor k, you just scale the side length of the original triangle by k, then build a new snowflake from that starting point.

All right, let's think about trying to scale the Koch snowflake in order to compute its dimension. Instead of dealing with the whole snowflake, let's just think about what's called the Koch curve, which is just one "side" of the Koch snowflake. To be a bit more precise, we just start with one side of the triangle, and then do our "tent-building" iterating process on that one line segment.

Ok, so we're dealing with the Koch curve instead, not a big deal. What can we do to figure out the dimension of this thing? We'll cleverly pick a scaling factor and use the trick we just talked about. I'm going to pick a scaling factor of $k = \frac{1}{3}$. What happens when I do this? Well, I then notice that if I take 4 copies of my scaled down Koch curve, I can cover the original Koch curve. That means my scaled down version is $\frac{1}{4}$ as big as my original. So my dimension equation is

$$\frac{m}{\widetilde{m}} = \frac{1}{4} = k^d = \left(\frac{1}{3}\right)^d$$

Note that I haven't made any attempt here to compute what m or $\widetilde{m}$ is. That would be much harder, but since I don't need to know, I won't bother. Anyway, back to my dimension equation. I can solve this using logarithms.

$$\frac{1}{4} = \left(\frac{1}{3}\right)^d \implies 4 = 3^d \implies \log 4 = d \log 3 \implies d = \frac{\log 4}{\log 3} \approx 1.2618590714\ldots$$

According to this calculation, the dimension of the Koch curve is about 1.26. Huh? That's very strange. I've only heard of shapes that have dimension 1, 2,3, etc. before. How could something have dimension which is not a whole number? Seeking the answer to that question has led to the whole theory of fractal geometry, so I can't get into it. But let's return to our definition.

Definition 2.7. Let X be a geometric shape with original measurement m (this could be length, area, volume, etc.). Let $k > 0$ be a positive real number (a "scaling factor"). Let $\widetilde{X}$ be the object X scaled by a factor of k, then let $\widetilde{m}$ be the measurement of $\widetilde{X}$. This satisfies the equation

$$\widetilde{m} = k^d m$$

The number d appearing in this equation is the **dimension** of X.

Nowhere in this definition does it require that d is a whole number. Most mathematical objects you are familiar with do, however, have a whole number dimension, according to this definition. But some don't. For example, the Koch curve, which we just said has dimension between 1 and 2.