Definition 1. Let R be a commutative ring with multiplicative identity. The **polynomial ring over** R is

$$R[x] = \{a_n x^n + a_{n-1} + \cdots + a_1 x + a_0 \mid a_n, \ldots, a_0 \in R\}$$

The **degree** of a polynomial is the highest power of x with a nonzero coefficient. The degree of the zero polynomial is $-\infty$. The **leading coefficient** is the coefficient of the highest degree term. A polynomial is **monic** if the leading coefficient is 1.

Definition 2. Let R_1 be a subring of a ring R_2. Then $R_1[x]$ is a subring of $R_2[x]$, so a polynomial $p(x) \in R_1[x]$ can also be viewed inside of $R_2[x]$. A **root** of $p(x)$ is an element of R_2 so that $p(a) = 0$.

Examples. Each polynomial below is an element of $\mathbb{Z}[x]$. Identify its degree, leading coefficient, whether it is monic, and any roots (and where those roots live).

(a) $p(x) = 2x + 1$

(b) $q(x) = x^2 - 5x + 6$

(c) A nonzero constant polynomial

(d) The zero constant polynomial

(e) A linear polynomial $p(x) = ax + b$ with $a, b \in \mathbb{Z}$

(f) $r(x) = x^2 - 2$

(g) $s(x) = x^2 + 1$

(h) A quadratic polynomial $p(x) = ax^2 + bx + c$ with $a, b, c \in \mathbb{Z}$

(i) $t(x) = x^3 - 1$

Proposition 1. *Let R be an integral domain. Then*

(a) For any $p, q \in R[x]$, we have $\deg(pq) = \deg(p) + \deg(q)$.

(b) $R[x]$ is an integral domain.

(c) A polynomial $p \in R[x]$ is a unit (of $R[x]$) if and only if p is a constant polynomial and that constant is a unit of R.

Proof.

$\square$

Proposition 2 (Polynomial long division)**.** *Let R be a commutative ring with unity. Let $f, g \in R[x]$ with $f \neq 0$, and assume that the leading coefficient of f is a unit (of R). Then there exist polynomials $q, r \in R[x]$ such that*

$$g(x) = q(x)f(x) + r(x)$$

and $\deg r < \deg f$*. Furthermore, if R is an integral domain, then q and r are unique.*

Proof.

$\square$

Corollary 1. *Let R be a ring and let $p \in R[x]$. An element $a \in R$ is a root of p if and only if $(x - a)$ is a factor of p, i.e. there exists $q \in R[x]$ so that $p(x) = (x - a)q(x)$.*